CHAPTER 1

THEORY OF ELASTICITY

1.1 A thin sheet made of an aluminum alloy having E = 67 GPa, G = 256 GPa and also v = 1/3 was used for two dimensional surface strain measurements. The measurements provided $\varepsilon_{xx} = 10.5x10^{-5}$, $\varepsilon_{yy} = -20x10^{-5}$, and $\gamma_{xy} = 240x10^{-5}$. Determine the corresponding stresses.

Solution:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix} = \frac{E}{(1-v^2)} \begin{bmatrix} \varepsilon_{xx} + v\varepsilon_{yy} \\ \varepsilon_{yy} + v\varepsilon_{xx} \end{bmatrix} = \begin{bmatrix} 2.89\ MPa \\ -12.44\ MPa \end{bmatrix}$$

$$\tau_{xy} = \frac{E\gamma_{xy}}{2(1+v)} = 60.45\ MPa$$

1.2 Determine **(a)** the principal stresses and strains and **(b)** the maximum shear stress for the case described in Problem 1.1.

Solution:

(a) $$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = -4.28\ MPa \pm 60.60\ MPa$$

$$\sigma_1 = 56.32\ MPa$$
$$\sigma_2 = -64.88\ MPa$$

The principal strains are

$$\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} + \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = -4.75x10^{-5} \pm 1.20x10^{-3}$$

$$\varepsilon_1 = 1.15x10^{-3}$$
$$\varepsilon_2 = -1.25x10^{-3}$$

(b) The maximum

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = 60.60\ MPa$$

1.3 Calculate the diameter of a 1-m long wire that supports a weight of 200 Newton. If the wire stretches 2 mm, calculate the strain and the stress induced by the weight. Let E = 207 GPa.

Solution:

$$\varepsilon = \frac{\Delta l}{l_o} = \frac{2\ mm}{10^3\ mm} = 2x10^{-3} = 0.20\%$$
$$d = \sqrt{\frac{4P}{\pi\sigma}} = 0.78\ mm$$
$$\sigma = \varepsilon E = 414\ MPa$$
$$P = A\sigma = (\pi/4)d^2\sigma = 200\ N$$

1.4 Derive an expression for the local uniform strain across the neck of a round bar being loaded in tension. Then, determine its magnitude if the original diameter is reduced 80%.

Solution:

Volume constancy:
$$AL = A_o L_o$$

$$\varepsilon_{zz} = \int_{L_o}^{L} \frac{dL}{L} = \ln(L/L_o) = \ln(d_o/d)^2$$
$$\varepsilon_{zz} = 2\ln(d_o/d)$$

If $d = 0.80 d_o$, then
$$\varepsilon_{zz} = 2\ln(1/0.80) = 0.4463$$
$$\varepsilon_{zz} = 44.63\%$$

1.5 The torsion of a bar containing a longitudinal sharp groove may be characterized by a warping function of the type [after F.A McClintock, Proc. Inter. Conf. on Fracture of Metals, Inst. of Mechanical Eng., London, (1956) 538] $w = \mu_z = \phi\int_0^r (ydx - xdy)$**.** The displacements are $\mu_x = 0$ and $\mu_y = rz\phi$, where ϕ and r are the angle of twist per unit length and the crack tip radius, respectively. The polar coordinates have the origin at the tip of the groove, which has a radius (R). Determine w, the shear strains γ_{rz} and $\gamma_{\theta z}$. In addition, predict the maximum of the shear strain $\gamma_{\theta z}$.

Solution:

Let $x = R\cos\theta$ and $y = R\sin\theta$ so that

$dx = -R\sin\theta d\theta$ and $dy = R\cos\theta d\theta$

$$\mu_z = \phi\int_0^r (ydx - xdy) = -\phi\int_0^\theta R^2 d\theta$$

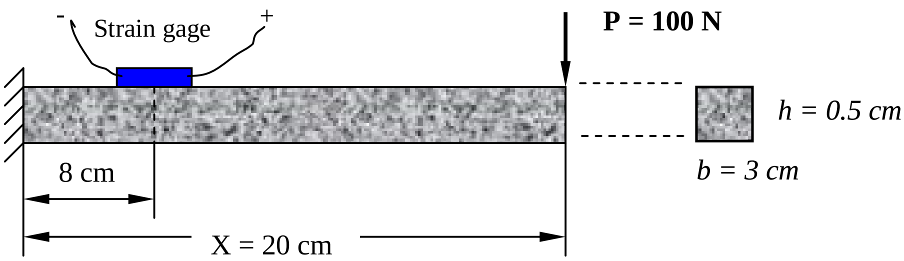
$$\gamma_{rz} = 0$$

$$\gamma_{\theta z} = \frac{\partial\mu_y}{\partial z} + \left(\frac{1}{r}\right)\frac{\partial\mu_z}{\partial\theta}$$

$$\gamma_{\theta z} = r\phi + \left(\frac{1}{r}\right)\left(-\phi R^2\right) = r\phi - \frac{\phi R^2}{r}$$

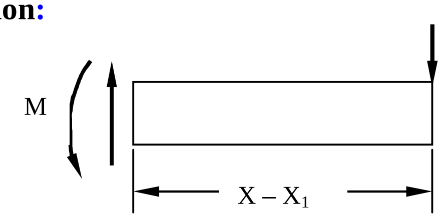
If $r \to 0$ then $\gamma_{\theta z} = \gamma_{\max} \to -\infty$

1.6 A cantilever beam having a cross-sectional area of 1.5 cm^2 is fixed at the left-hand side and loaded with a 100 Newton downward vertical force at the extreme end as shown in the figure shown below. Determine the strain in the strain gage located at 8 cm from the fixed end of the shown steel cantilever beam. The steel modulus of elasticity of is E = 207 GPa.



Solution:

<u>FBD:</u>



$$\sum M^{+}\downarrow = 0 \quad (Moments)$$

$$P(X - X_1) - M = 0$$

$$M = P(X - X_1)$$

$$\sigma = \frac{MC}{I}$$

$$C = \frac{h}{2}$$

$$I = \frac{bh^3}{12}$$

Thus,

$$\sigma = \frac{P(X - X_1)(h/2)}{\frac{nh^3}{12}} = \frac{6P(X - X_1)}{bh^2}$$

Using Hooke's law yields the elastic strain

$$\varepsilon = \frac{\sigma}{E}$$

$$\varepsilon = \frac{6P(X - X_1)}{Ebh^2} = \frac{(6)(100\ N)(20\ m - 8\ m)x10^{-2}}{(207x10^6\ N/m)(3x10^{-2}\ m)(0.50x10^{-2}\ m)^2} = 4.64x10^{-4}$$

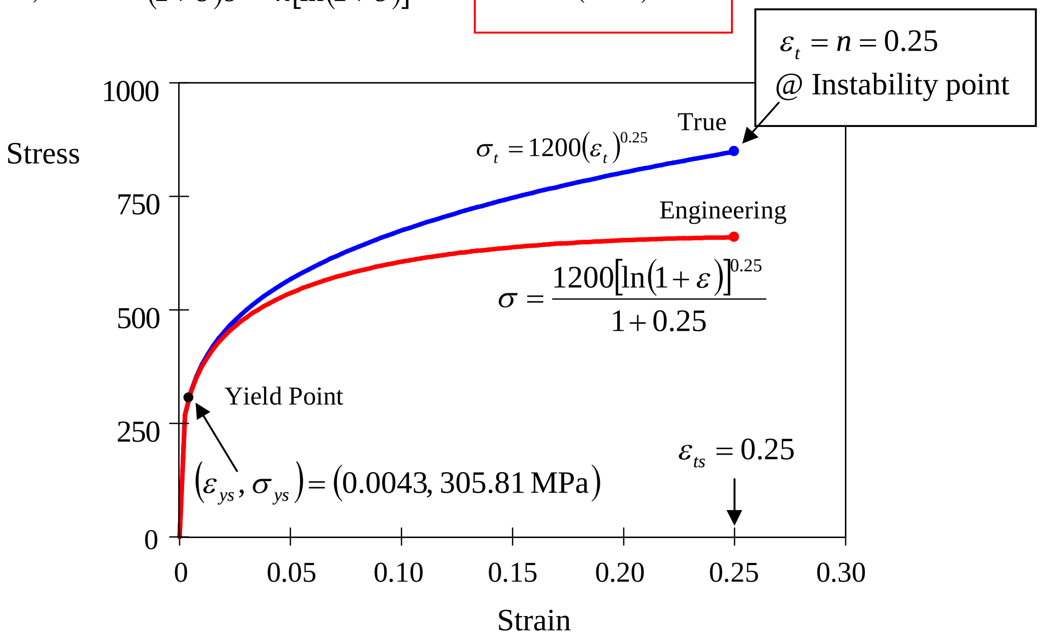
1.7 The stress-strain behavior of an annealed low-carbon steel (σ_{ys} = 200 MPa and E = 207 GPa) obeys the Hollomon equation with k = 530 MPa and n =0.25. **(a)** Plot the true and engineering stress-strain curves. Calculate **(b)** the tensile strength (σ_{ts}) and **(c)** the strain energy density up to the instability point.

Solution:

(a) Plots

$$\sigma_t = (1+\varepsilon)\sigma \qquad \boxed{\sigma_t = k(\varepsilon_t)^n} \qquad \boxed{\sigma = \frac{k[\ln(1+\varepsilon)]^n}{(1+\varepsilon)}}$$

$$\varepsilon_t = \ln(1+\varepsilon) \qquad (1+\varepsilon)\sigma = k[\ln(1+\varepsilon)]^n$$



(b) The engineering yield strain and the true strain at the instability point on the true stress-strain curve are, respectively

$$\varepsilon_{ys} = \frac{\sigma_{ys}}{E} = \frac{900\ MPa}{207x10^3\ MPa} = 4.3478\times10^{-3}$$

$$\varepsilon_t = n = 0.25 \quad @ \text{ Instability point}$$

At the instability or ultimate tensile strength point (maximum),

$$\sigma_{\max} = k\varepsilon_t^n = (1{,}200\ MPa)(0.25)^{0.25} = \boxed{848.53\ MPa}$$

$$\varepsilon_t = \ln(1+\varepsilon)$$

$$\varepsilon = -1+\exp(\varepsilon_t) = -1+\exp(0.25) = \boxed{0.28403}$$

$$\sigma_{\max} = (1+\varepsilon)\sigma_{ts}$$
$$\sigma_{ts} = \sigma_{\max}/(1+\varepsilon)$$
$$\sigma_{ts} = (374.77\ MPa)/(1+0.28403) = \boxed{291.87\ \text{MPa}}$$

(c) The strain energy density (*W*) is just the area under the stress-strain curve. In general, the true and engineering strain energy densities are

$$W_t = \left(\int_0^{\varepsilon_{ys}} \sigma d\varepsilon\right)_{Elastic} + \left(\int_{\varepsilon_{ys}}^{\varepsilon_{ts}} \sigma d\varepsilon\right)_{Plastic}$$
$$W_t = \int_0^{\varepsilon_{ys}} E\varepsilon d\varepsilon + \int_{\varepsilon_{ys}}^{\varepsilon_{ts}} k\varepsilon^n d\varepsilon = \frac{E\varepsilon_{ys}^2}{2} + \frac{k}{1-n}\left[(\varepsilon_{ts})^{1+n} + (\varepsilon_{ys})^{1+n}\right]$$
$$W_t = \frac{(207{,}000)(4.3478\times 10^{-3})^2}{2} + \frac{1{,}200}{1-0.25}\left[(0.28403)^{1+0.25} + (4.3478\times 10^{-3})^{1+0.25}\right] = 335.50\ MPa$$
$$\boxed{W_t = 335.50\ \frac{MN}{m^2}\frac{m}{m} = 335.50\ MJ/m^3}$$

and

$$W = \left(\int_0^{\varepsilon_{ys}} \sigma d\varepsilon\right)_{Elastic} + \left(\int_{\varepsilon_{ys}}^{\varepsilon_{ts}} \sigma d\varepsilon\right)_{Plastic}$$
$$W = \int_0^{\varepsilon_{ys}} E\varepsilon d\varepsilon + \int_{\varepsilon_{ys}}^{\varepsilon_{ts}} \sigma = \frac{k[\ln(1+\varepsilon)]^n}{(1+\varepsilon)}\ d\varepsilon = \frac{E\varepsilon_{ys}^2}{2} + \frac{k}{n+1}\left[\ln(\varepsilon_{ts}+1-\varepsilon_{ys}-1)\right]^{n+1}$$
$$W = \frac{(207{,}000)(4.3478\times 10^{-3})^2}{2} + \left(\frac{1{,}200}{0.25+1}\right)\left[\ln(0.28403 - 4.3478\times 10^{-3})\right]^{0.25+1}$$
$$W = 0.99932 - 0.95718i$$

Therefore, the engineering strain energy density can not be determined analytically using the modified Hollomon equation. Instead, the solution can be achieved numerically. Thus,

$$W = \int_0^{0.0043478} 207000\varepsilon d\varepsilon + \int_{0.0043478}^{0.28403} \frac{1200[\ln(1+\varepsilon)]^{0.25}}{(1+\varepsilon)}\ d\varepsilon = 170.60\ MPa$$
$$\boxed{W = 170.60\ MJ/m^3}$$

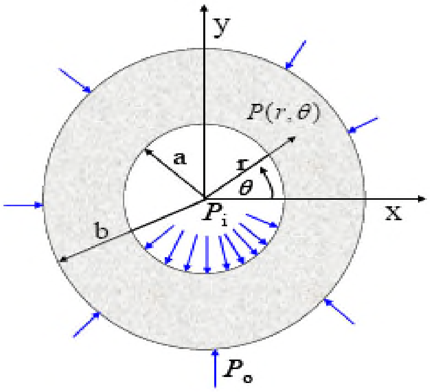
$$W_t \approx 2W$$

1.8 The figure below shows a schematic cross-sectional view of a pressure vessel (hollow cylinder) subjected to internal and external pressures. Determine the stresses at a point $P(r,\theta)$ in polar coordinates when **(a)** $P_i \neq 0$ and $P_i \neq 0$, **(b)** $P_o = 0$, **(c)** $P_i = 0$ and **(d)** $a = 0$ so that the hollow cylinder becomes a solid cylinder. **(e)** Plot $\sigma_{rr} = f(r)$ and $\sigma_{rr} = f(r)$. Let $a = 450\ mm$, $b = 800\ mm$, $P_i = 0$ and $P = 40\ MPa$. The valid radius range must be $0.45\ m \leq r \leq 0.80\ m$. Use the following Airy's stress function

$$\phi = c_1 + c_2 \ln r + c_3 r^2$$

Along with the boundary conditions

$$\sigma_{rr} = -P_i \qquad \tau_{r\theta} = 0 \qquad @\ r = a$$
$$\sigma_{rr} = -P_o \qquad \tau_{r\theta} = 0 \qquad @\ r = b$$



Solution:

(a) Derivatives of $\phi = c_1 + c_2 \ln r + c_3 r^2$

$$\frac{\partial \phi}{\partial r} = \frac{c_2}{r} + 2c_3 r \qquad \frac{\partial \phi}{\partial \theta} = 0 \qquad \frac{\partial^2 \phi}{\partial r \partial \theta} = 0$$
$$\frac{\partial^2 \phi}{\partial r^2} = -\frac{c_2}{r^2} + 2c_3 \qquad \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

From eq. (1.58),

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{c_2}{r^2} + 2c_3$$
$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = -\frac{c_2}{r^2} + 2c_3$$
$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} = 0$$

Using the boundary conditions yields

$$\sigma_{rr} = -P_i = \frac{c_2}{r^2} + 2c_3 \qquad -P_i = \frac{c_2}{r^2} + 2c_3$$

$$\sigma_{rr} = -P_o = \frac{c_2}{r^2} + 2c_3 \qquad -P_o = \frac{c_2}{r^2} + 2c_3 \tag{a}$$

Replace r for the proper radius and solve eq. (a) for c_2 and c_3

$$-P_i = \frac{c_2}{r^2} + 2c_3$$

$$-P_o = \frac{c_2}{r^2} + 2c_3$$

from which

$$-P_i = \frac{c_2}{a^2} + 2c_3$$

$$2c_3 = -P_i - \frac{c_2}{a^2}$$

and

$$-P_o = \frac{c_2}{b^2} + 2c_3$$

$$2c_3 = -P_o - \frac{c_2}{b^2}$$

Then,

$$P_i + \frac{c_2}{a^2} = P_o + \frac{c_2}{b^2} \qquad 2c_3 = -P_o - \frac{c_2}{b^2} = -P_o - \frac{1}{b^2}\frac{a^2b^2}{b^2 - a^2}\left(P_o - P_i\right) = -P_o + \frac{a^2b^2}{b^2 - a^2}\left(P_i - P_o\right)$$

$$\frac{c_2}{a^2} - \frac{c_2}{b^2} = P_o - P_i \qquad \& \qquad 2c_3 = \frac{a^2P_i}{b^2 - a^2} - \frac{a^2P_i}{b^2 - a^2} - P_o = \frac{a^2P_i}{b^2 - a^2} - P_o\left(\frac{a^2}{b^2 - a^2} + 1\right)$$

$_o - P_i$

$$c_2 = \frac{a^2b^2}{b^2 - a^2}\left(P_o - P_i\right) \qquad 2c_3 = \frac{a^2P_i}{b^2 - a^2} - \frac{b^2P_o}{b^2 - a^2}$$

$$c_2 = \frac{a^2b^2}{b^2 - a^2}\left(P_o - P_i\right) \qquad \& \qquad c_3 = \frac{a^2P_i - b^2P_o}{2\left(b^2 - a^2\right)}$$

Thus,

$$\sigma_{rr} = \frac{a^2P_i - b^2P_o}{b^2 - a^2} + \frac{a^2b^2\left(P_o - P_i\right)}{\left(b^2 - a^2\right)r^2}$$

$$\sigma_{\theta\theta} = \frac{a^2P_i - b^2P_o}{b^2 - a^2} - \frac{a^2b^2\left(P_o - P_i\right)}{\left(b^2 - a^2\right)r^2}$$

$$\tau_{r\theta} = 0$$

(b) If $P_o = 0,$ then

$$\sigma_{rr} = \frac{a^2 P_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{a^2 P_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$\tau_{r\theta} = 0$$

(c) If $P_i = 0,$ then

$$\sigma_{rr} = \frac{b^2 P_o}{b^2 - a^2}\left(\frac{a^2}{r^2} - 1\right)$$

$$\sigma_{\theta\theta} = \frac{b^2 P_o}{b^2 - a^2}\left(\frac{a^2}{r^2} + 1\right)$$

$$\tau_{r\theta} = 0$$

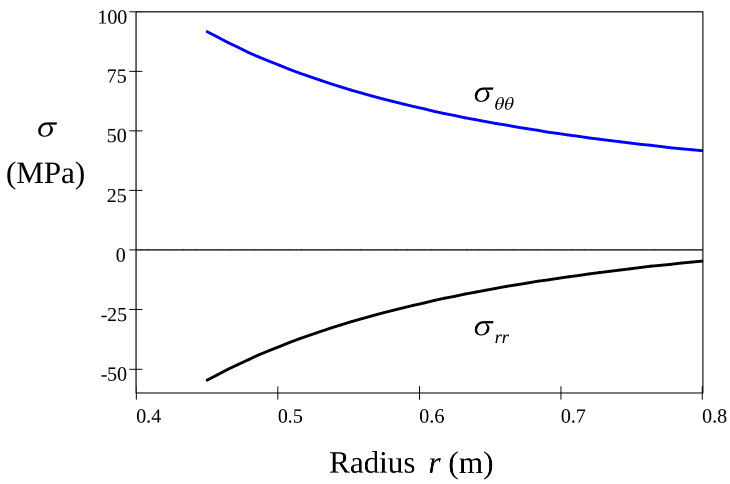
(d) If $a = 0,$ then

$$\sigma_{rr} = -P_o$$

$$\sigma_{\theta\theta} = -P_o$$

$$\tau_{r\theta} = 0$$

(e) The plot

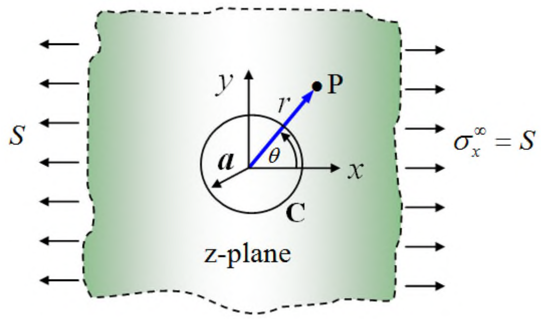


1.9 Consider an infinite plate with a central hole subjected to a remote uniform stress as shown in Example 1.5. The boundary conditions for this loaded plate are (1) $\sigma_x^\infty = \sigma_x = S$ and $\sigma_y = \tau_{xy} = 0$ at $r = \infty$ and (2) $\sigma_r = \tau_{r\theta} = 0$ at $r = a$. Use the following complex potentials [12]

$$\psi'(z) = \frac{S}{4}\left(1 - \frac{2a^2}{z^2}\right) \text{ and } \chi''(z) = -\frac{S}{2}\left(1 - \frac{a^2}{z^2} + \frac{3a^4}{z^4}\right)$$

to determine σ_r, σ_θ and $\tau_{r\theta}$ (in polar coordinates).

Solution: The infinite plate with a central hole subjected to a remote uniform stress is



Use the following stress expressions

$$\sigma_r + \sigma_\theta = 2\psi'(z) + 2\overline{\psi}'(\bar{z}) = 4\,\mathrm{Re}\,\psi'(z)$$

$$\sigma_\theta - \sigma_r + i2\tau_{r\theta} = 2[\bar{z}\psi''(z) + \chi''(z)]e^{i2\theta}$$

Recall that the Euler's formula for z and z² are

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$\bar{z} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

$$z^m = r^m e^{im\theta} = r^3(\cos m\theta + i\sin m\theta)$$

and their real and imaginary parts are

$$\mathrm{Re}\,z = r\cos\theta \;\&\; \mathrm{Re}\,z^m = r^m\cos m\theta$$

$$\mathrm{Im}\,z = r\sin\theta \;\&\; \mathrm{Im}\,z^m = r^m\sin m\theta$$

Apply the Euler's formula to the above complex potentials so that

$$\psi'(z) = \frac{S}{4}\left(1 - \frac{2a^2}{z^2}\right) = \frac{S}{4}\left(1 - \frac{2a^2}{r^2e^{i2\theta}}\right)$$

$$\psi'(z) = \frac{S}{4}\left(1 - \frac{2a^2}{r^2}e^{-i2\theta}\right)$$

$$4\,\mathrm{Re}\,\psi'(z) = S\left(1 - \frac{2a^2}{r^2}\cos 2\theta\right)$$

(a) The first stress equation:

$$\sigma_r + \sigma_\theta = 4\,\mathrm{Re}\,\psi'(z) = S\left(1 - \frac{2a^2}{r^2}\cos 2\theta\right) \qquad \text{(a)}$$

Then,

$$\psi'(z) = \frac{S}{4} - \frac{a^2}{2z^2}$$
$$\psi''(z) = +\frac{2Sa^2}{2z^3} = \frac{Sa^2}{z^3} = \frac{Sa^2}{r^3 e^{i3\theta}} = \frac{Sa^2}{r^3} e^{-i3\theta}$$
$$\psi''(z) = \frac{Sa^2}{r^3}\cos 3\theta \text{ (Real part)}$$

Thus,

$$\bar{z}\psi''(z) = (re^{-i\theta})\left(\frac{Sa^2}{r^3 e^{i3\theta}}\right) = \frac{Sa^2}{r^2} e^{-i4\theta}$$

(b) The second potential:

$$\chi''(z) = -\frac{S}{2}\left(1 - \frac{a^2}{z^2} + \frac{3a^4}{z^4}\right) = -\frac{S}{2}\left(1 - \frac{a^2}{r^2 e^{i2\theta}} + \frac{3a^4}{r^4 e^{i4\theta}}\right)$$

Hence,

$$\sigma_\theta - \sigma_r + i2\tau_{r\theta} = 2[\bar{z}\psi''(z) + \chi''(z)]e^{i2\theta}$$
$$\sigma_\theta - \sigma_r + i2\tau_{r\theta} = 2\left[\frac{Sa^2}{r^2} e^{-i4\theta} e^{i2\theta} - \frac{S}{2} e^{i2\theta} + \frac{Sa^2 e^{i2\theta}}{2r^2 e^{i2\theta}} - \frac{3Sa^4 e^{i2\theta}}{2r^4 e^{i4\theta}}\right]$$
$$\sigma_\theta - \sigma_r + i2\tau_{r\theta} = S\left[\frac{2a^2}{r^2} e^{-i2\theta} - e^{i2\theta} + \frac{a^2}{r^2} - \frac{3a^4}{r^4} e^{-i2\theta}\right] \tag{b}$$

Real parts:

$$\sigma_\theta - \sigma_r = S\left[\frac{2a^2}{r^2}\cos 2\theta - \cos 2\theta + \frac{a^2}{r^2} - \frac{3a^4}{r^4}\cos 2\theta\right] \tag{c}$$

Add eqs. (a) and (c) and solve for σ_θ

$$\sigma_\theta = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{d1}$$

Use eq. (a) to solve for σ_r

$$\sigma_r = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{d2}$$

Using the imaginary part of eq. (b) yields shear stress $\tau_{r\theta}$ as

$$i2\tau_{r\theta} = S\left(i\frac{2a^2}{r^2}\sin 2\theta - i\sin 2\theta + i\frac{3a^4}{r^4}\sin 2\theta\right)$$
$$\tau_{r\theta} = -\frac{S}{2}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin 2\theta \tag{d3}$$

At $r = a$

$$\sigma_r = 0 \tag{f1}$$
$$\sigma_\theta = S - 2S\cos 2\theta = S(1 - 2\cos 2\theta) \tag{f2}$$
$$\tau_{r\theta} = 0 \tag{f3}$$

1.10 Use the Cauchy-Riemann condition to show that (a) $f(z) = 1/z$ is analytic and (b) its derivative is $f(z)= -1/z^2$.

Solution:

(a) If $z = x + iy$, then

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$$\frac{1}{z} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

Let

$$u = \frac{x}{x^2+y^2} \qquad v = -\frac{y}{x^2+y^2}$$

Then

$$\frac{\partial u}{\partial x} = \frac{\left(x^2+y^2\right)x' - x\left(x^2+y^2\right)'}{\left(x^2+y^2\right)^2} = \frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{\left(x^2+y^2\right)^2}$$

and

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{x^2-x^2}{(x^2+y^2)^2}$$

Thus,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Therefore, $f(z) = 1/z$ is analytic because $\partial u/\partial y = \partial v/\partial x$.

(b) The derivative of $f(z) = 1/z$ is

$$f'(z) = \frac{d}{dz}\left(\frac{1}{z}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} + i\frac{2xy}{\left(x^2+y^2\right)^2} = \frac{(y+ix)^2}{\left(x^2+y^2\right)^2}$$

$$= \frac{-i(-iy+x)^2}{i\left(x^2+y^2\right)^2} = \frac{-(x-iy)^2}{\left(x^2+y^2\right)^2} = -\frac{\bar{z}^2}{[(x+iy)(x-iy)]^2}$$

$$= -\frac{\bar{z}^2}{[z\bar{z}]^2} = -\frac{1}{z^2}$$

1.12 Evaluate the Cauchy integral formula given below for the complex function when $z_o = \pi$.

$$f(z_o) = \frac{1}{2\pi i}\int \frac{\cos z}{z^2-1}dz$$

Solution:
The integral has to be modified as

$$f(z_o) = \frac{1}{2\pi i}\int \frac{\cos z}{z^2-1}dz$$

$$f(z_o) = \frac{1}{2\pi i}\int \frac{\cos z}{(z+1)(z-1)}dz$$

$$f(z_o) = \frac{1}{}\int \frac{f(z)}{z-1}dz$$

$$\int \frac{f(z)}{z-1}dz = 2\pi i f(z_o)$$

where

$$f(z) = \frac{\cos z}{z+1}$$

Then,

$$\int \frac{f(z)}{z-1}dz = 2\pi i f(z_o)$$

$$\int \frac{f(z)}{z-1}dz = 2\pi i \left| \frac{\cos z_o}{z_o+1} \right|_{z_o=\pi}$$

$$\int \frac{f(z)}{z-1}dz = 2\pi i \left[\frac{\cos \pi}{\pi+1} \right] = 1.52i$$

CHAPTER 2

INTRODUCTION TO FRACTURE MECHANICS

2.1 Show that the applied stress $\sigma \to 0$ as the crack tip radius $\rho \to 0$. Explain

Solution:

From eq. (2.28),

$$\sigma_{\max} = 2\sigma\sqrt{\frac{a}{\rho}} \qquad \textit{and} \qquad \sigma = \frac{1}{2}\sigma_{\max}\sqrt{\frac{\rho}{a}}$$

Therefore, $\sigma \to 0$ as $\rho \to 0$. This means that the applied stress $\sigma \to 0$ since the material has fractured or separated, at least into two pieces, and there is not a crack present so that $\rho \to 0$. On the other hand, $\rho \to 0$ means that the existing crack is very sharp as in the case of a fatigue crack.

2.2 For a Griffith crack case, crack propagation takes place if the strain energy satisfies the inequality $[U(a) - U(a + \Delta a)] \geq 2\Delta a\gamma$, where $\Delta a =$ crack extension and γ is the surface energy. Show that the crack driving force or the strain energy release rate at instability is $G \geq \frac{\partial U(a)}{\partial a}$.

Solution:

Use the Taylor's series to expand the left side term of the inequality. Thus,

$$U(a + \Delta a) = U(a) + \frac{1}{1!}\frac{\partial U(a)}{\partial a}\Delta a + \frac{1}{2!}\frac{\partial^2 U(a)}{\partial a^2}\Delta a^2 + \frac{1}{3!}\frac{\partial^3 U(a)}{\partial a^3}\Delta a^3 + \ldots\ldots\ldots\ldots\ldots \tag{2.1}$$

Using the first two terms yields

$$U(a + \Delta a) = U(a) + \frac{\partial U(a)}{\partial a}\Delta a \tag{2.2}$$

$$U(a) - U(a + \Delta a) = -\frac{\partial U(a)}{\partial a}\Delta a \tag{2.2}$$

Then,

$$-\frac{\partial U(a)}{\partial a}\Delta a \geq 2\Delta a\gamma \tag{2.3}$$

$$-\frac{\partial U(a)}{\partial a} \geq 2\gamma = G \tag{2.4}$$

Therefore,

$$G \geq \frac{\partial U(a)}{\partial a} \tag{2.5}$$

2.3 A 1mmx15mmx100mm steel strap has a 3-mm long central crack is loaded to failure. Assume that the steel is brittle and has $E = 207{,}000\ MPa$, $\sigma_{ys} = 1500\ MPa$, and $K_{IC} = 70\ MPa\sqrt{m}$. Determine the critical stress (σ_c) and the critical strain energy release rate.

Solution:
$a = 1.5\ mm$ and $a / w = 0.10$

The geometry correction factor:

$$\alpha = f(a/w) = \sqrt{\frac{w}{\pi a}\tan\left(\frac{\pi a}{w}\right)} = 1.07$$

The stress intensity factor

$$K_{IC} = \alpha\sigma_f\sqrt{\pi a}$$

$$\sigma_c = \frac{K_{IC}^2}{\alpha\sqrt{\pi a}} \approx 1020\ MPa$$

The strain energy release rate

$$G_{IC} = \frac{K_{IC}^2}{E}$$

$$G_{IC} = 23.70\ kPa\cdot m$$

$$G_{IC} = 23.70\ kN/m^2$$

2.4 Suppose that a structure made of plates has one cracked plate. If the crack reaches a critical size, will that plate fracture or the entire structure collapse? Explain.

Answer: If the structure does not have crack stoppers, the entire structure will collapse since the cracked plate will be the source of structural instability.

2.5 What is crack instability based on according to Griffith criterion?

Answer: It is based on the energy consumption creating new surfaces or in developing crack extension. This energy is the surface energy 2γ.

2.6 Can the Griffith Theory be applied for a quenched steel containing 1.2%C, if a penny-shaped crack is detected?

Answer: Despite the quenched steel is brittle, the Griffith Theory applies only if a plastic zone does not form at the crack tip.

2.7 Will the Irwin Theory (or modified Griffith Theory) be valid for a changing plastic zone size as the crack advances?

Answer: No. It is valid if the plastic zone remains constant as it the case in many materials under plane strain conditions.

2.8 What are the major roles of the surface energy and the stored elastic energies in a crack growth situation?

Answer: The surface energy acts as a retarding force for crack growth and the stored elastic energy acts to extend the crack. Thus, concurrent with crack growth is the recovery of elastic strain energy by the relaxation of atomic bonds above and below the fracture plane.

2.9 What does happen to the elastic strain energy when crack growth occur?

Answer: It is released as the crack driving force for crack growth.

2.10 What does ∂U/∂a = 0 mean?

Answer: Crack growth occurs at $\sigma = \sigma_f$, as shown in eq. (2.23), since the magnitude of crack growth and the crack resistance force become equal. That is,

$$2\gamma_s = \frac{\pi a \beta \sigma^2}{E}$$

If $2\gamma_s > \dfrac{\pi a \beta \sigma^2}{E}$, then fracture occurs and the fracture toughness without any plastic zone deformation is $G_c = 2\gamma_s$ at $\sigma = \sigma_f$, as shown in eq. (2.31) with $\gamma_p = 0$. Thus, $G_c = 2\gamma_s$ is related to an irreversible process of fracture.

2.11 Derive eq. (2.28) starting with eq. (2.20).

Solution:

From eq. (2.20),

$$K_t = \frac{\sigma_{\max}}{\sigma} = 2\sqrt{\frac{a}{\rho}}$$

$$\frac{1}{2}\sigma_{\max} = \sqrt{\frac{a}{\rho}}\,\sigma$$

Multiplying this expression by $\sqrt{\pi\rho}$ yields the stress intensity factor as per eq. (2.28)

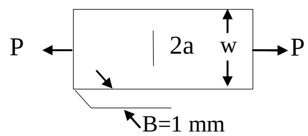
$$\frac{1}{2}\sigma_{\max}\sqrt{\pi\rho} = \sqrt{\frac{a}{\rho}}\,\sigma\sqrt{\pi\rho} = \sigma\sqrt{\pi a}$$

$$K_I = \sigma\sqrt{\pi a}$$

CHAPTER 3

LINEAR ELASTIC FRACTURE MECHANICS

3.1 A steel strap 1-mm thick and 20-mm wide with a through-the-thickness central crack 4-mm long is loaded to failure. **(a)** Determine the critical load if K_{IC} = 80 MPa$\sqrt{m}$ for the strap material. **(b)** Use the available correction factor, $\alpha = f(a/w)$, for this crack configuration. Now, do the following three comparisons to have a better understanding: Feddersen: $\sigma_c = (fraction)\sigma_\infty$, Irwin: $\sigma_c = (fraction)\sigma_\infty$ and Koiter-Benthem: $\sigma_c = (fraction)\sigma_\infty$.



Solution:
B = 1mm, w = 20mm, 2a = 4 mm, *a*/*w* = 0.10, K_{IC} = 80 MPa$\sqrt{m}$, $K_I = \sigma_c\sqrt{\pi a}\cdot f(a/w)$,

(a) Finite plate:

From eqs. (3.31) and (3.32) along with $a/w = 0.10\ rad. = (180^o)(0.10)/\pi = 5.7296^o$

Feddersen [12]: $f\left(\frac{a}{w}\right) = \sqrt{\sec\left(\frac{\pi a}{w}\right)} = 1.025$ $\qquad \sigma_c = 984.64\,MPa$

Irwin [13]: $f\left(\frac{a}{w}\right) = \sqrt{\frac{w}{\pi a}\tan\left(\frac{\pi a}{w}\right)} = 1.017$ $\qquad \sigma_c = 992.38\,MPa$

Koiter-Benthem [14]: $f\left(\frac{a}{w}\right) = \frac{1}{\sqrt{1-2\frac{a}{w}}}\left[1-\left(\frac{a}{w}\right)+1.304\left(\frac{a}{w}\right)^2\right] = 1.021$ $\qquad \sigma_c = 988.49\,MPa$

(b) Infinite plate approach [*a*/*w* $\to$ 0 since *f*(a/*w*) $\to$ 1].

$$\sigma_c = \frac{K_{IC}}{\sqrt{\pi a}} = \frac{80MPa\sqrt{m}}{\sqrt{\pi\left(2\times10^{-3}\,m\right)}} = 1009.25MPa$$

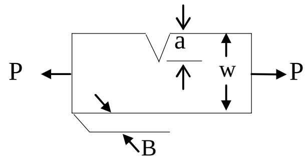
Feddersen: $\sigma_c \cong 97.56\%\cdot\sigma_c$ (infinite)

Irwin: $\sigma_c \cong 98.33\%\cdot\sigma_c$ (infinite)

Koiter: $\sigma_c \cong 97.94\%\cdot\sigma_c$ (infinite)

Therefore, the calculated σ_c values do not differ much since the plate dimensions are large enough and the initial crack size is relatively small.

3.2 A steel tension bar 8-mm thick and 50-mm wide with an initial single-edge crack of 10-mm long is subjected to a uniaxial stress σ = 140 MPa. **(a)** Determine the stress intensity factor K_I. If K_{IC} = 60 MPa$\sqrt{m}$, is the crack stable? **(b)** Determine the critical crack size, and **(c)** determine the critical load.



Solution:

σ = P/A= 140 MPa, K_{IC} = 60 MPa$\sqrt{m}$, $K_I = \sigma\sqrt{\pi a}\cdot f\left(\frac{a}{w}\right)$

B = 8 mm, w = 50 mm, a = 10 mm, $x = a/w = 0.20$, A = Bw

From Table 3.1, $K_I = \alpha\sigma\sqrt{\pi a}$ and

$$\alpha = f(x) = 1.12 - 0.231(x) + 10.55(x)^2 - 21.71(x)^3 + 30.38(x)^4$$
$$\alpha = f(x = 0.20) = 1.37$$

(a) The applied stress intensity factor

K_I = 34 MPa$\sqrt{m}$
K_{IC} = 60 MPa$\sqrt{m}$

Therefore, the crack is stable because $K_I < K_{IC}$.

(b) $K_{IC} = \sigma\sqrt{\pi a_c}\,.\,f(x)$

$$a_c = \frac{1}{\pi}\left[\frac{K_{IC}}{f(x)\cdot\sigma}\right]^2 = 0.0311\,m = 31.10\,mm$$

(c) The critical load

$$\sigma = \frac{P}{A}$$
$$A = Bw = (8\ mm)(50\ mm) = 400\ mm^2 = 4\times10^{-4}\ m^2$$
$$P = A\sigma = \left(4\times10^{-4}\,m^2\right)\left(140\frac{MN}{m^2}\right) = 0.056MN = 56\ kN \quad \text{(Applied)}$$
$$P_c = \frac{AK_{IC}}{f(x)\sqrt{\pi a}} = \frac{\left(4\times10^{-4}\right)\left(60\frac{MN}{m^2}\sqrt{m}\right)}{1.37\sqrt{\pi\left(10\times10^{-3}m\right)}} = 0.0988\ MN = \underline{\underline{98.84\ kN}} \quad \text{(Critical)}$$

3.3 A very sharp penny-shaped crack with a diameter of 22-mm is completely embedded in a highly brittle solid. Assume that catastrophic fracture occurs when a stress of 600 MPa is applied. a) What is the fracture toughness for this solid? (Assume that this fracture toughness is for plane strain conditions). b) If a sheet 5-mm thick of this solid is prepared for fracture-toughness testing. Would the fracture-toughness value [(calculated in part a)] be an acceptable number according to the ASTM E399 standard? Use σ_{ys} = 1342 MPa. c) What thickness would be required for the fracture-toughness test to be valid?

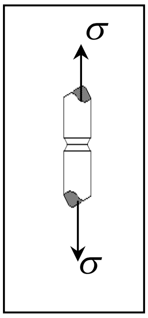
Solution:

a) $K_{IC} = \frac{2}{\pi}\sigma\sqrt{\pi a} = \frac{2}{\pi}(600\ MPa)\sqrt{\pi(11\times 10^{-3}\ m)} = 71\ MPa\sqrt{m}$

b) $B \geq 2.5\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = 2.5\left(\frac{71\ MPa\sqrt{m}}{1342\ MPa}\right)^2 = 7\ mm$ > 5 mm (Not valid)

c) $B \geq 7\ mm$

3.4 (a) One guitar steel string has a miniature circumferential crack of 0.009 mm deep. This implies that the radius ratio is almost unity, $d/D \approx 1$. b) Another string has a localized miniature surface crack (single-edge crack like) of 0.009 mm deep. Assume that both strings are identical with an outer diameter of 0.28 mm. If a load of 49 N is applied to the string when being tuned, will it break? Given properties: $K_{IC} = 15 MPa\sqrt{m}, \sigma_{ys} = 795 MPa.$



Solution:

(a) Circumferential crack

If $a = \frac{D-d}{2} = 0.009\ mm$, then $d = 0.262\ mm$

$$\sigma = \frac{4P}{\pi d^2} = \frac{(4)(49N)}{\pi(0.262x10^{-3}\ m)} = 908.87\ MPa$$

$$\alpha = f(d/D) = \frac{1}{2}\sqrt{\frac{D}{d}}\left[\frac{1}{2} + \frac{D}{d} + \frac{3}{8}\left(\frac{d}{D}\right) - 0.36\left(\frac{d}{D}\right)^2 + 0.73\left(\frac{d}{D}\right)^3\right]$$

$For\ d/D = 0.9357,$
$\alpha = 1.14$

$$K_I = \alpha\sigma\sqrt{\pi a}$$
$$K_I = (1.14)(908.87\ MPa)\sqrt{\pi\left(0.009\times10^{-3}\,m\right)}$$
$$K_I = 5.51\,MPa\sqrt{m}$$

(b) Surface edge crack

$$\alpha = 1.12$$
$$K_I = \alpha\sigma\sqrt{\pi a}$$
$$K_I = (1.12)(908.87\ MPa)\sqrt{\pi\left(0.009\times10^{-3}\,m\right)}$$
$$K_I = 5.41\,MPa\sqrt{m}$$

Therefore, the strings will not break since $K_I < K_{IC}$ in both cases. When $d/D \approx 1$ and the crack size is very small, either approach gives similar results.

3.5 A 7075-T6 aluminum alloy is loaded in tension. Initially the 10mmx100mmx500mm plate has a 4-mm single-edge through–the-thickness crack. **(a)** Is this test valid? **(b)** Calculate the maximum allowable tension stress this plate can support, **(c)** Is it necessary to correct K_I due to crack-tip plasticity? Why? or Why not? **(d)** Calculate the design stress and stress intensity factor if the safety factor is 1.5. Data: σ_{ys} = 586 MPa and K_{IC} = 33 MPa$\sqrt{m}$.

Solution:

(a) $B \geq 2.5\left(K_{IC}/\sigma_{ys}\right)^2 = 7.93\,mm$ and $a/w = 0.0375$

Plane strain condition holds because the actual thickness is greater than the required value.

(b) $K_I = \alpha\sigma\sqrt{\pi a}$; a = 4 mm and w = 100 mm

$$\alpha = f\left(a/w\right) = 1.12 - 0.231\left(\frac{a}{w}\right) + 10.55\left(\frac{a}{w}\right)^2 - 21.71\left(\frac{a}{w}\right)^3 + 30.38\left(\frac{a}{w}\right)^4$$
$$\alpha = f\left(a/w\right) = f(0.04) = 1.13$$

Thus,

$$\sigma = \frac{K_{IC}}{\alpha\sqrt{\pi a}} = 261\,MPa$$

(c) Plastic zone correction:

$$r \approx \frac{1}{6\pi}\left(K_{IC}/\sigma_{ys}\right)^2 \approx 0.17mm$$

$$K_{eff} = \alpha\sigma\sqrt{\pi a_{eff}} \quad \text{with} \quad a_{eff} = a + r = 4.17 \text{ mm}$$

$$K_{eff} = (1.13)(261\ MPa)\sqrt{\pi\left(4.17x10^{-3}\ m\right)}$$

$$K_{eff} = 33.76\ MPa\sqrt{m}$$

Therefore, there is no need for crack tip plasticity since $K_{I,eff} \approx K_{IC}$.

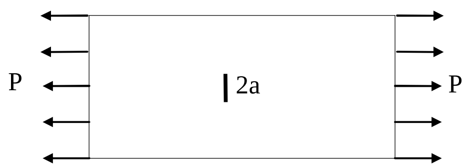
(d) $\sigma_d = \dfrac{\sigma}{SF} = \dfrac{261\ MPa}{1.5} = 174\ MPa$

$$K_{Id} = \frac{K_{IC}}{SF} = \frac{33\ MPa\sqrt{m}}{1.5} = 22\ MPa\sqrt{m}$$

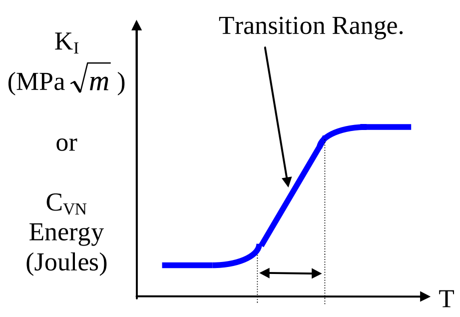
3.6 A steel ship deck (30mm thick, 12m wide, and 20m long) is stressed in the manner shown below. It is operated below its ductile-to-brittle transition temperature (with K_{Ic} = 28.3 MPa $\sqrt{m}$). If a 65-mm long through–the-thickness central crack is present, calculate the tensile stress for catastrophic failure. Compare this stress with the yield strength of 240 MPa for this steel.

Solution:

2a = 65mm
w = 12 m
L = 20m



$\alpha = f(a/w) \rightarrow 1$ since a/w $\rightarrow 0$



$$K_{IC} = \alpha\sigma\sqrt{\pi a} \qquad K_{IC} = 28.3\ MPa\sqrt{m} \text{ and } \sigma_{ys} = 240 MPa$$

$$\sigma = \frac{K_{IC}}{\sqrt{\pi a}} = \frac{28.3\ MPa\sqrt{m}}{\sqrt{\pi\left(\frac{65}{2}\times 10^{-3}\ m\right)}} = 88.57\ MPa$$

Therefore, $\sigma < \sigma_{ys}$ and to assure structural integrity, the safety factor (SF) that can be used to avoid fracture or crack propagation is in the order of

$$SF \geq \frac{\sigma_{ys}}{\sigma} = \frac{240\ MPa}{88.57\ MPa} = 2.71$$

3.7 Show that $\frac{K_I}{da} > 0$ for crack instability in a large plate under a remote tensile external stress.

Solution:

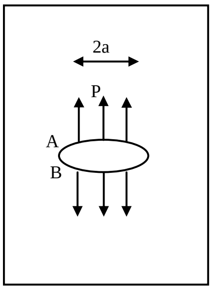
$$K_I = \sigma\sqrt{\pi a}$$

$$\frac{dK_I}{da} = \frac{1}{2}\sigma\sqrt{\frac{\pi}{a}} > 0 \text{ since } \sigma > 0 \text{ and } a > 0$$

3.8 The plate below has an internal crack subjected to a pressure P on the crack surface. The stress intensity factors at points A and B are

$$K_A = \int \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}}\cdot dx$$

$$K_B = \int \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}}\cdot dx$$



Use the principle of superposition to show that the total stress intensity factor is of the form $K_I = P\sqrt{\pi a}$

Solution:
According to the principle of superposition, the total stress intensity factor is

$$K_I = K_A + K_B = \frac{P}{\sqrt{\pi a}}\int\left(\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}}\right)dx = 2P\sqrt{\frac{a}{\pi}}\cdot\int\frac{dx}{\sqrt{a^2 - x^2}}$$

Furthermore,

$$\sqrt{\frac{a+x}{a-x}}+\sqrt{\frac{a-x}{a+x}}=\frac{\sqrt{a+x}}{\sqrt{a-x}}+\frac{\sqrt{a-x}}{\sqrt{a+x}}=\frac{(a+x)+(a-x)}{\sqrt{(a-x)(a+x)}}=\frac{2a}{\sqrt{a^2-x^2}}$$

$$\int\frac{dx}{\sqrt{a^2-x^2}}=ar\,c\sin\frac{x}{a}$$

and

$$K_I=2P\sqrt{\frac{a}{\pi}}\arcsin\frac{x}{a}$$

If $x = a$ on the crack surface, then $\arcsin(1)=\pi/2=90^o$ and

$$K_I=2P\sqrt{\frac{a}{\pi}}ar\sin\frac{x}{a}=\frac{2P}{\pi}\sqrt{\pi a}\cdot\frac{\pi}{2}=P\sqrt{\pi a}$$

3.9 A pressure vessel is to be designed using the **leak-before-break** criterion based on the circumferential wall stress and plane strain fracture toughness. The design stress is restricted by the yield strength σ_{ys} and a safety factor (S_F). Derive expressions for **(a)** the critical crack size and **(b)** the maximum allowable pressure when the crack size is equals to the vessel thickness.

Solution:

(a) The critical crack size

$$K_{IC}=\alpha\sigma\sqrt{\pi a_c}$$

$$\sigma_d=\frac{\sigma_{ys}}{S_F}$$

Thus,

$$K_{IC}=\frac{\alpha\sigma_{ys}}{S_F}\sqrt{\pi a_c}$$

$$a_c=\frac{1}{\pi}\left(\frac{S_F}{\alpha}\right)^2\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2$$

Select some alloy having known K_{IC} and σ_{ys} values. The alloy with the highest $\frac{K_{IC}^2}{\sigma_{ys}}$ and $\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2$ values should be used for constructing the pressure vessel.

3.10 A stock of steel plates with G_{IC} = 130 kJ/m², σ_{ys} = 2,200 MPa, E = 207 GPa, v = 1/3 are used to fabricate a cylindrical pressure vessel (d_i = 5 m and B = 25.4 mm). The vessel fractured at a pressure of 20 MPa. Subsequent failure analysis revealed an internal semi-elliptical surface crack of a = 2.5 mm and $2c$ = 10 mm. **(a)** Use a fracture mechanics approach to predict the critical crack length this steel would tolerate. **(b)** Based on this catastrophic failure, another vessel was constructed with d_o = 5.5 m and d_i = 5 m. Will this new vessel fracture at a pressure of 20 MPa if there is an internal semi-elliptical surface crack having the same dimensions as in part **(a)**?

Solution:

(a) The vessel is based on the thin-wall theory since the thickness is

$$B \le \frac{d_i}{20} = \frac{5x10^3\ mm}{20} = 250\ mm \quad \text{(Requirement)}$$

$B = 25.4\ mm$ (Given) is less than the required thickness. So use the thin-wall theory.

Thus, the fracture hoop stress is

$$\sigma_h = \frac{P_i d_i}{2B} = \frac{(20\ MPa)(5x10^3\ mm)}{(2)(25.4\ mm)}$$

$$\sigma_h = 1{,}968.50\ MPa$$

Using eq. (2.34) yields the critical crack length

$$a_c = \frac{EG_{IC}}{\pi(1-v^2)\sigma_h^2}$$

$$a_c = \frac{(207x10^3\ MPa)(130x10^{-3}\ MPa.m)}{\pi(1-1/3^2)(1{,}968.50\ MPa)^2}$$

$$a_c = 2.49\ mm \approx 2.5\ mm$$

(b) For the new vessel,

$$B = \frac{d_o - d_i}{2} = \frac{5.6\ m - 5\ m}{2} = 0.30\ m$$

$$\frac{d_o}{d_i} = \frac{5.6}{5} = 1.12$$

Also,

$$\sigma_h = \left[\frac{(d_o/d_i)^2 + 1}{(d_o/d_i)^2 - 1}\right] P_i = 8.86 P_i$$

$$\sigma_h = 8.86(20\ MPa) = 177.23\ MPa$$

$$\sigma = \sigma_h + P_i = 9.86 P_i = 197.23\ MPa$$

Then,

$$a/2c = 2.5/10 = 0.25$$

$$\sigma/\sigma_{ys} = 197.23/2,200 = 0.09$$

From eq. (3.44),

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]^2 - \frac{7}{33}\left(\frac{\sigma}{\sigma_{ys}}\right)^2$$

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + (0.25)^2\right]^2 - \frac{7}{33}(0.09)^2$$

$$Q = 1.63$$

Thus, the applied stress intensity factor is

$$K_I = \sigma\sqrt{\frac{\pi a}{Q}}$$

$$K_I = \left(197.23\ MPa\right)\sqrt{\frac{\pi(2.5x10^{-3})}{1.63}}$$

$$K_I = 13.69\ MPa\sqrt{m}$$

From eq. (2.33) at fracture,

$$K_{IC} = \sqrt{\frac{EG_{IC}}{1-v^2}}$$

$$K_{IC} = \sqrt{\frac{\left(207x10^3\ MPa\right)\left(130x10^{-3}\ MPa.m\right)}{1-(1/3)^2}}$$

$$K_{IC} = 174\ MPa\sqrt{m}$$

Therefore, the new vessel will not fracture because $K_I < K_{IC}$.

3.11 A cylindrical pressure vessel with B = 25.4 mm and d_i = 800 mm is subjected to an internal pressure P_i. The material has $K_{IC} = 31\ MPa\sqrt{m}$ and σ_{ys} = 600 MPa. **(a)** Use a safety factor to determine the actual pressure P_i. **(b)** Assume there exists a semi-elliptical surface crack with a = 5 mm and $2c$ = 25 mm and that a pressure surge occurs causing fracture of the vessel. Calculate the fracture internal pressure P_f. **(c)** Calculate the critical crack length.

Solution:

(a) Let's figure out which pressure vessel wall theory should be used.

$$B \preceq \frac{d_i}{20} = \frac{800\ mm}{20} = 40\ mm$$

Therefore, the thin-wall theory should be used. The actual or applied pressure is

$$\sigma_h = \frac{P_i d_i}{2B} = \frac{\sigma_{ys}}{S_F}$$

$$P_i = \frac{2B\sigma_{ys}}{S_F d_i} = \frac{2(25.4\ mm)(600\ MPa)}{2.5(800\ mm)}$$

$$P_i = 15.24\ MPa$$

(b) Using the principle of superposition yields

$$\sigma = \sigma_h + P_i = \frac{P_i d_i}{2B} + P_i$$

$$\sigma = \left(\frac{d_i}{2B} + 1\right)P_i = \left(\frac{800}{2x25.4} + 1\right)P_i$$

$$\sigma = 16.75P_i = 16.75(15.24\ MPa)$$

$$\sigma = 255.24\ MPa \ \text{(Actual)}$$

Also,

$$\frac{a}{2c} = \frac{5}{25} = 0.20 \qquad \frac{a}{B} = \frac{5}{25.4} = 0.20$$

$$\frac{\sigma}{\sigma_{ys}} = \frac{255.24}{600} = 0.43 \qquad M = M_k = 1 \text{ [See eq.(3.46) \& Figure 3.6]}$$

From eq. (3.44),

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]^2 - \frac{7}{33}\left(\frac{\sigma}{\sigma_{ys}}\right)^2$$

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + (0.2)^2\right]^2 - \frac{7}{33}(0.43)^2$$

$$Q = 1.50$$

From eq. (3.41),

$$K_{IC} = MM_k\sigma\sqrt{\frac{\pi a}{Q}} = 16.75P_f\sqrt{\frac{\pi a}{Q}}$$

Now, the surge pressure is

$$P_f = \frac{K_{IC}}{16.75}\sqrt{\frac{Q}{\pi a}} = \frac{31\ MPa\sqrt{m}}{16.75}\sqrt{\frac{1.50}{\pi(5x10^{-3}\ m)}}$$

$$P_f = 18.09\ MPa$$

(c) The critical crack length is

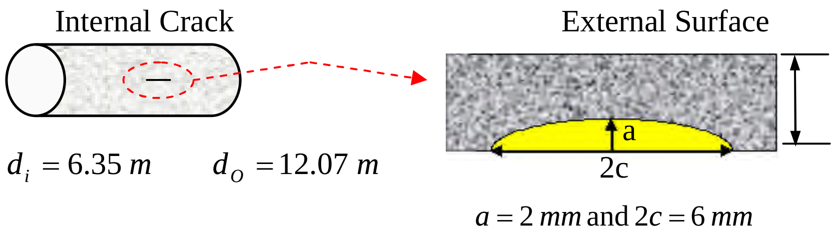
$$K_{IC} = \sigma\sqrt{\frac{\pi a}{Q}} = 16.75P_i\sqrt{\frac{\pi a_c}{Q}}$$

$$a_c = \frac{Q}{\pi}\left(\frac{K_{IC}}{16.75P_i}\right)^2 = \frac{1.5}{\pi}\left(\frac{31\ MPa\sqrt{m}}{16.75\times 15.24\ MPa}\right)^2$$

$$a_c = 7.04\ mm$$

3.12 This is a problem that involves strength of materials and fracture mechanics. An AISI 4340 steel is used to design a cylindrical pressure having an inside diameter and an outside diameter of 6.35 cm and 12.07 cm, respectively. The hoop stress is not to exceed 80% of the yield strength of the material. **(a)** Is the structure a thin-wall vessel or a thick-wall pipe? **(b)** What is the internal pressure? **(c)** Assume that an internal semi-elliptical surface crack exist with a = 2 mm and $2c$ = 6 mm. Will the vessel fail? (**d**) Will you recommend steel for the pressure vessel? Why? or Why not? **(e)** What is the maximum crack length the AISI 4340 steel can tolerate? Explain.

Solution:



From Table 3.2, σ_{ys} = 1,476 MPa and K_{IC} = 81 MPa for AISI 4340 steel.

(a) Using the diameters yields

$$\frac{d_o}{d_i} = \frac{12.07}{6.35} = 1.09$$

$$B = \frac{d_o - d_i}{2} = \frac{12.07\ cm - 6.35\ cm}{2}$$

$$B = 2.86\ cm$$

Therefore, the pressure vessel is based on the thick-wall theory because

$$\frac{d_o}{d_i} \simeq 1.1$$

$$B > \frac{d_i}{20} = 0.32$$

(b) From eq. (3.43e),

$$\sigma_h = \left[\frac{(d_o/d_i)^2+1}{(d_o/d_i)^2-1}\right]P_i = 0.8\sigma_{ys}$$

$$P_i = 0.8\sigma_{ys}\left[\frac{(d_o/d_i)^2-1}{(d_o/d_i)^2+1}\right]$$

Thus, the internal pressure is

$$P_i = 0.8\left(1{,}476\ MPa\right)\left[\frac{(1.09)^2 - 1}{(1.09)^2 + 1}\right] =$$

$$P_i = 101.51\ MPa$$

(c) Fracture mechanics:

$$\sigma = \sigma_h + P_i = 0.8\sigma_{ys} + P_i$$

$$\sigma = 0.8(1{,}476\ MPa) + 101.51\ MPa$$

$$\sigma = 1{,}282.30\ MPa$$

Thus, $a/2c = 2/6 = 0.33$, $\sigma/\sigma_{ys} = 0.87$ and the shape factor becomes

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]^2 - \frac{7}{33}\left(\frac{\sigma}{\sigma_{ys}}\right)^2$$

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + (0.33)^2\right]^2 - \frac{7}{33}(0.87)^2$$

$$Q = 1.66$$

Then,

$$K_I = \sigma\sqrt{\frac{\pi a}{Q}} = \left(1{,}282.30\ MPa\right)\sqrt{\frac{\pi\left(2x10^{-3}\ m\right)}{1.66}}$$

$$K_I = 78.89\ MPa\sqrt{m}$$

Therefore, the pressure vessel will not fail because $K_I < K_{IC}$.

(d) Denote that K_I is slightly below the K_{IC} value and safety precautions should be taken because a minor increase in pressure due to a pressure surge will cause fracture. The margin of safety is very small and therefore, a new steel should be used having a higher K_{IC} value.

(e) The critical crack length is

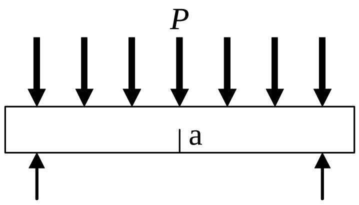
$$K_{IC} = \sigma\sqrt{\frac{\pi a_c}{Q}}$$

$$a_c = \frac{Q}{\pi}\left(\frac{K_{IC}}{\sigma}\right)^2 = \frac{1.66}{\pi}\left(\frac{81\ MPa\sqrt{m}}{1{,}282.30\ MPa}\right)$$

$$a_c = 2.11\ mm$$

This result implies that the pressure vessel may fracture because the existing crack of 2 mm in length is approximately 5% smaller than its critical value. Therefore, a new steel should be used to design the pressure vessel having the same dimensions and being subjected to the design conditions.

3.13 A simple supported beam made of soda glass (E =71 MPa and G_{IC} = 12 J/m²) is subjected to a uniformly distributed bending load as shown in the figure below. Assume an initial crack length of 0.1 mm due to either stress corrosion cracking (SCC) or a mechanical defect introduced during fabrication or handling. The design stress (working stress) and the crack velocity equation are 0.10 MPa and $v = \beta K_I^m = (6.24\ m/s)K_I^{15.83}$, respectively. Use this information to calculate the lifetime of the beam if the maximum bending stress is given by σ = (MB/2)/I and M = $\sigma_d L^2/8$, where the second moment of area is $I = wB^3/12$.



$B = 15\ mm$
$w = 200\ mm$
$L = 2\ m$

Solution:

Given data: $G_I = 12\ J/m^2 = 12\ Pa.m,\ \ E = 71\, GPa$
$B = 20\ mm,\ \ w = 200\ mm,\ \ L = 2\ m$

The applied stress

$$\sigma = \frac{3\sigma_d L^2}{4wB^2} = \frac{(3)(0.10\ MPa)(2{,}000\ mm)^2}{(4)(200\ mm)(15\ mm)^2} = 6.67\ MPa$$

Using Griffith expression, eq. (2.34), gives the critical crack length

$$a_c = \frac{EG_{IC}}{\pi\sigma^2} = \frac{(71x10^3\ MPa)(12x10^{-6}\ MPa.m)}{(\pi)(6.67\ MPa)^2} = 6.10\ mm$$

This is a large crack length for a brittle material such as soda glass. The applied and the critical stress intensity factors, eq. (3.29), are

$$K_I = \alpha\sigma\sqrt{\pi a} = (1.12)(6.67\ MPa)\sqrt{\pi(10^{-4}\ m)} = 0.13\, MPa\sqrt{m}$$
$$K_{IC} = \alpha\sigma\sqrt{\pi a_c} = (1.12)(6.67\ MPa)\sqrt{\pi(6.10x10^{-3}\ m)} = 1.03\, MPa\sqrt{m}$$

The crack velocity equation can be rearranged using the Chain Rule. Thus,

$$v = \frac{da}{dt} = \frac{da}{dK_I}\frac{dK_I}{dt}$$

But,

$$K_I = \alpha\sigma\sqrt{\pi a}$$

$$a = \frac{K_I^2}{\pi\alpha^2\sigma^2}$$

Then,

$$\frac{da}{dK_I} = \frac{2K_I}{\pi\alpha^2\sigma^2}$$

and

$$v = \frac{da}{dt} = \frac{2K_I}{\pi\alpha^2\sigma^2}\frac{dK_I}{dt} \quad \& \quad v = \beta K_I^m = (6.24\ m/s)K_I^{15.83}$$

$$\beta K_I^m = \frac{2K_I}{\pi\alpha^2\sigma^2}\frac{dK_I}{dt}$$

Rearranging this equation and integrating with respect to time yields

$$dt = \frac{2K_I}{\pi\alpha^2\sigma^2\beta K_I^m}dK_I$$

$$\int_o^{t_f} dt = \frac{2}{\pi\alpha^2\sigma^2\beta}\int_{K_I}^{K_{IC}} K_I^{1-m}dK_I$$

where $\int_{K_I}^{K_{IC}} K_I^{1-m}dK_I = -\frac{1}{m-2}K^{2-m}\Big|_{K_I}^{K_{IC}} = -\frac{1}{m-2}\left(K_{IC}^{2-m} - K_I^{2-m}\right)$

Then,

$$t_f = -\frac{2}{\pi\alpha^2\sigma^2\beta}\frac{1}{m-2}\left(K_{IC}^{2-m} - K_I^{2-m}\right)$$

$$t_f = -\left[\frac{2}{(\pi)(1.12)^2(6.67\ MPa)^2(6.24\ m/s)}\right]\left(\frac{1}{15.83-2}\right)\left[(1.03)^{2-15.83} - (0.13)^{2-15.83}\right]$$

$$t_f = 2.3733x10^8 \text{ seconds} = 7.52\ \textit{years}$$

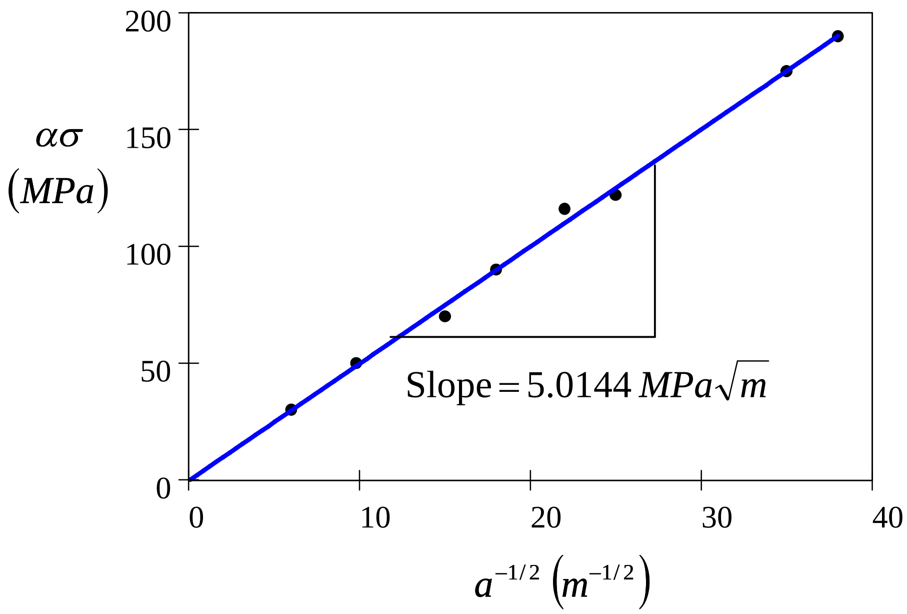
Therefore, the beam will last 7.53 years prior to fracture.

3.14 Plot the given data for a hypothetical solid SE(T) specimen and determine the plane strain fracture toughness K_{IC}. Here, $\alpha = f(a/w)\sqrt{\pi}$ is the modified geometry correction factor.

$a^{-1/2}\ (m^{-1/2})$	6	9.8	15	18	22	25	35	38
$\alpha\sigma\ (MPa)$	30	50	70	90	116	122	175	190

Solution:

The plot



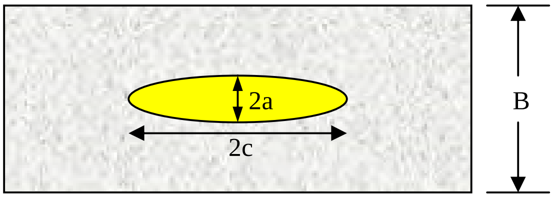
Linear curve fitting gives

$\alpha\sigma = 5.0144a^{-1/2} - 0.42832$

$$Slope = \frac{\Delta(\alpha\sigma)}{\Delta(a^{-1/2})} = 5.0144\ MPa\sqrt{m}$$

$$K_{IC} = \alpha\sigma\sqrt{\pi a} \approx 5.0144\ MPa\sqrt{m}$$

3.15 A pressure vessel ($L >> B = 15$ mm, $d_i = 2$ m) is to be made out of a weldable steel alloy having $\sigma_{ys} = 1{,}200$ MPa and $K_{IC} = 85$ MPa. If an embedded elliptical crack (2a = 5 mm and 2c = 16 mm) as shown below is perpendicular to the hoop stress, due to welding defects, the given data correspond to the operating room temperature and the operating pressure is 8 MPa, then calculate the applied stress intensity factor. Will the pressure vessel explode?



Solution:

$$d_o = d_i + 2B = 2\ m + 2 \times 15 \times 10^{-3}\ m = 2.03\ m$$

$$\frac{d_o}{d_i} = \frac{2.03}{2} = 1.02$$

$$B \leq \frac{d_i}{20} = \frac{2 \times 10^3\ mm}{20} = 100\ mm$$

The pressure vessel is of thin-wall type because $d_o/d_i < 1.1$ and $B \leq d_i/20$. The hoop stress and the stress intensity factor for the embedded elliptical crack are, respectively

$$\sigma_h = \frac{P_i d_i}{2B}$$

$$K_I = \frac{\sigma_h \sqrt{\pi a}}{\Phi}$$

where

$$\Phi = \frac{\pi}{4}\left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]$$

Then,

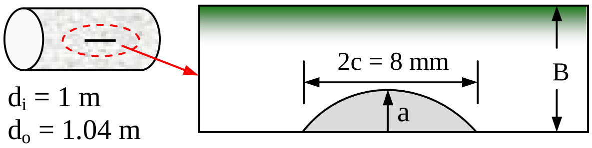
$$K_I = \frac{4 P_i d_i \sqrt{\pi a}}{2\pi B\left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]}$$

$$K_I = \frac{4\left(8\ MPa\right)\left(2\ m\right)\sqrt{\pi\left(2.5 \times 10^{-3}\ m\right)}}{2\pi\left(15 \times 10^{-3}\ m\right)\left[\frac{3}{4} + \left(\frac{2.5}{16}\right)^2\right]}$$

$$K_I = 77.71\ MPa\sqrt{m}$$

The pressure vessel will not explode because $K_I < K_{IC}$.

3.16 A steel pressure vessel subjected to a constant internal pressure of 20 MPa contains an internal semi-elliptical surface crack with dimensions shown below. Calculate K_I when **(a)** Q and σ/σ_{ys} in eq. (3.44) and **(b)** $Q = 1 + 1.464(a/c)^{1.65}$ as per reference [31]. **(c)** Compare results and explain. **(d)** If fracture does not occur, calculate the safe factors $S_F^{(a)}$ and $S_F^{(b)}$. Explain the results. Data: $K_{IC} = 92\ MPa\sqrt{m}$, $\sigma_{ys} = 900\ MPa$, $v = 0.3$ and $B = 30\ mm$.



Solution:

For $d_o/d_i = 1.04$ and $B = (d_o - d_i)/2 = (1.04\ m - 1\ m)/2 = 0.02\ m = 20\ mm$

$P_i = 20\ MPa$, $a/2c = 2/8 = 1/4$

(a) $\sigma = \sigma_h + P_i = \dfrac{P_i d_i}{2B} + P_i = \left(1 + \dfrac{d_i}{2B}\right)P_i = \left(1 + \dfrac{1\ m}{2x0.02\ m}\right)(20\ MPa) = 520\ MPa < \sigma_{ys}$

$\sigma/\sigma_{ys} = 520/900 = 0.57778$

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]^2 - \frac{7}{33}\left(\frac{\sigma}{\sigma_{ys}}\right)^2$$

$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + \left(\frac{1}{4}\right)^2\right]^2 - \frac{7}{33}(0.57778)^2 = 1.5581$$

$$K_I = 1.12\sigma\sqrt{\frac{\pi a}{Q}} = 1.12(520\ MPa)\sqrt{\frac{\pi(2x10^{-3})}{1.5581}} = 36.98 MPa\sqrt{m}$$

Validity as per ASTM E399

$$B_{ASTM} \geq 2.5\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = 2.5\left(\frac{92\ MPa\sqrt{m}}{900\ MPa}\right)^2 = 2.6123x10^{-2}\ m = 26.12\ mm$$

$B = 30_{mm} > B_{ASTM}$ *OK!*

(b) For $Q = 1 + 1.464(a/c)^{1.65} = 1 + 1.464(2/4)^{1.65} = 1.4665$,

$$K_I = 1.12\sigma\sqrt{\frac{\pi a}{Q}} = 1.12(520\ MPa)\sqrt{\frac{\pi(2x10^{-3})}{1.4665}} = 38.12\ MPa\sqrt{m}$$

(c) Comparing these results yields $K_I^{(a)} \approx 0.97 K_I^{(b)}$ and the percent error is

$$\%Error = \frac{38.12 - 36.98}{38.12} x100 \approx 3\%$$

This indicates that the Q equations give dissimilar results despite the different mathematical approaches being used. Nonetheless, both calculated K_I values are below K_{IC} and therefore, fracture does not occur.

(d) Since $K_I < K_{IC}$ the safety factor can be calculated. Thus,

$$S_F^{(a)} = K_{IC} / K_I^{(a)} = 92/36.98 = 2.49$$
$$S_F^{(b)} = K_{IC} / K_I^{(b)} = 92/38.12 = 2.41$$

These are reasonably high values and therefore, the pressure vessel can be kept in service, but a nondestructive technique must be used to monitor any crack growth that may lead to $a \to a_c$ and $K_I \to K_{IC}$.

3.17 An aluminum-alloy plate has a plane strain fracture toughness of 30 MPa$\sqrt{m}$. Two identical single-edge cracked specimens are subjected to tension loading. **(a)** One specimen having a 2-mm crack fractures at a stress level of 330 MPa. Calculate the geometry correction factor $\alpha = f(a/w)$. **(b)** Will the second specimen having a 1-mm crack fracture when loaded at 430 MPa?

Solution:

(a) The geometry correction factor is

$$\alpha = f(a/w) = \frac{K_{IC}}{\sigma\sqrt{\pi a}} = \frac{30\ MPa\sqrt{m}}{(330\ MPa)\sqrt{\pi(2x10^{-3}\ m)}} = 1.1469$$

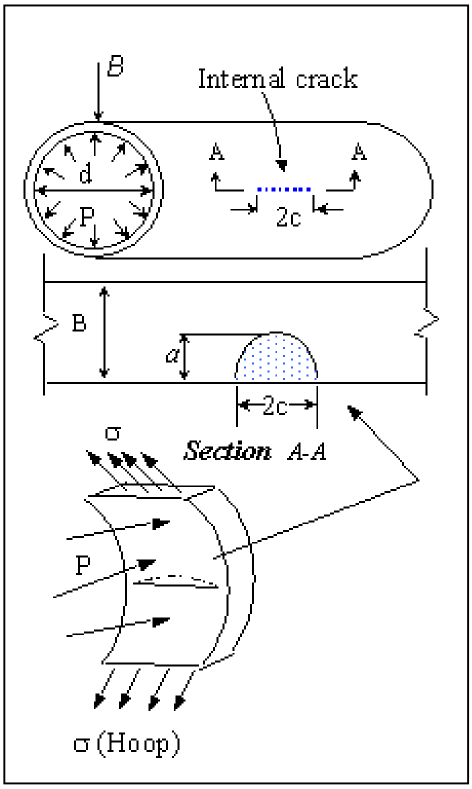
(b) The stress intensity factor for the second specimen is

$$K_I = \alpha\sigma\sqrt{\pi a} = (1.1469)(430\ MPa)\sqrt{\pi(1x10^{-3}\ m)} = 27.64\ MPa\sqrt{m}$$

Therefore, fracture will not occur because $K_I < K_{IC}$.

EXTRA PROBLEM NOT INCLUDED IN THE BOOK

A thin-walled cylindrical pressure vessel contains gas at a pressure of 100 MPa. During the initial pressurization of the vessel, the axial and tangential strains were measured on the inside surface as $\varepsilon_{xx} = 500\text{x}10^{-6}$ and $\varepsilon_{yy} = 750\text{x}10^{-6}$, respectively. Calculate **a**) the axial and hoop stresses associated with these strains. Assume that an undetected internal semi-elliptical surface crack (a = 1 mm deep and 4 mm long) grows 0.5 mm/year and that the pressure remains constant. **b**) Deduce whether or not the vessel will fracture by calculating the applied stress intensity factor using the constant hoop stress (tangential stress) and **c**) if the vessel will not fracture, then calculate the time it takes for such unpleasant occurrence.



Note: Only the shape factor should be used to correct the stress intensity factor K_I.

Given data: $\sigma_{ys} = 600\ MPa$

$K_{IC} = 30\ MPa\sqrt{m}$

$E = 207\ GPa$

$v = 0.3$

$B = 2\ cm$

$d = 0.50\ m$ (Diameter)

Solution:

a) From eq. (1.6) along with $\sigma_{zz} = P$, the inner strains are

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} \sigma_{xx} - \nu(\sigma_{yy} + P) \\ \sigma_{yy} - \nu(\sigma_{xx} + P) \end{bmatrix} \tag{1.6}$$

Solving these two equations simultaneously yields

$$\sigma_{xx} = \frac{E}{1-\nu}\left(\epsilon_{yy} + \frac{\epsilon_{xx}-\epsilon_{yy}}{1+\nu}\right) + \frac{\nu P}{1-\nu} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy}) + \frac{\nu P}{1-\nu} \tag{a}$$

$$\sigma_{yy} = E\epsilon_{yy} + \nu(\sigma_{xx} - P) \tag{b}$$

Given data:

$$\epsilon_{xx} = 500 \times 10^{-6} \qquad \epsilon_{yy} = 750 \times 10^{-6} \qquad E = 207{,}000\ MPa$$
$$\sigma_{ys} = 600\ MPa \qquad K_{IC} = 30\ MPA\sqrt{m}$$

Thus,

$$\sigma_{xx} = \frac{207{,}000\ MPa}{1-0.3^2}\left(500 \times 10^{-6} + (0.3)\left(750 \times 10^{-6}\right)\right) - \left(\frac{0.3}{1-0.3}\right)(100\ MPa)$$

$\sigma_{xx} = 122.06\ MPa$ (Axial Stress)

$$\sigma_{yy} = (207{,}000\ MPa)\left(750 \times 10^{-6}\right) + (0.3)(122.06\ MPa - 100\ MPa)$$

$\sigma_{yy} = 161.87\ MPa$ (Hoop Stress or Tangential Stress)

b) ASTM E399 thickness requirement, eq. (3.30):

$$a, B_{ASTM} \geq 2.5 \times 10^3 \left(\frac{30}{600}\right)^2 = 6.25\ mm$$

$B > B_{ASTM}$, but a is not met for use LEFM.

The valid average tangential or hoop stress regardless of the wall thickness is

$$\sigma_{av} = \sigma_{yy,av} = \frac{Pd}{2B} = \frac{100 \times 0.5 \times 10^2\ cm}{2 \times 2\ cm} = 1{,}250\ MPa$$

The applied stress intensity at initial pressurization hoop stress is

$$E = 207\ GPa \qquad K_{IC} = 30\ MPa\sqrt{m} \qquad \nu = 0.3$$
$$\sigma = \sigma_{yy} = 161.87\ MPa \qquad \sigma_{ys} = 600\ MPa$$
$$\sigma/\sigma_{ys} = 161.87/600 = 0.26978$$
$$a = 1\ mm \qquad 2c = 4\ mm \qquad a/2c = 0.25$$
$$Q = \left(\frac{\pi}{2}\right)^2\left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]^2 - \frac{7}{33}\left(\frac{\sigma}{\sigma_{ys}}\right)^2 = 1.6134$$
$$K_I = \sigma\sqrt{\pi a/Q} = (161.87\ MPa)\sqrt{\frac{\pi\left(1 \times 10^{-3}\ m\right)}{1.6134}} = 7.14\ MPa\sqrt{m}$$

- The the initial pressurization process of the vessel will not cause fracture because $K_I < K_{IC}$ at $\sigma = \sigma_{yy} = 161.87\ MPa$.

- If $a = B = 20\ mm$ (leak-before-fracturecondition), then

  $K_I = \sigma_{yy}\sqrt{\pi a/Q} = (161.87\ MPa)\sqrt{\frac{\pi(20\times10^{-3}\ m)}{1.6134}} = 31.94\ MPa\sqrt{m}$

  Therefore, fracture will occur because $K_I = 31.94\ MPa\sqrt{m} > K_{IC}$

- When $\sigma_{yy} = \sigma_{av}$ fracture will occur because

  $K_I = \sigma_{av}\sqrt{\pi a/Q} = (1,250\ MPa)\sqrt{\frac{\pi(1\times10^{-3}\ m)}{1.6134}} = 55.16\ MPa\sqrt{m}$

  $K_I > K_{IC} = 30\ MPa\sqrt{m}$

c) If the tangential stress, $\sigma_{yy} = 161.87\ MPa$, is assumed constant thoughtout the test, then the critical crack length becomes

$$K_{IC} = \sigma\sqrt{\pi a_c/Q}$$

$$a_c = \frac{Q}{\pi}\left(\frac{K_{IC}}{\sigma_{yy}}\right)^2 = \frac{1.6134}{\pi}\left(10^3\right)\left(\frac{30}{161.87}\right)^2 = 17.64\ mm$$

$$a_c \lesssim B = 20\ mm$$

Thus, the life of the vessel is approximately

$$t = (a_c - a)/(da/dt) = (17.64\ mm - 1\ mm)/(0.5\ mm/year)$$

$$t = 33.28\ years$$

This is an unrealistic result because $\sigma_{yy} = \sigma_{av}$ after the initial pressurization process is completed. Consequently, the vessel fractures immediately and its lifetime is basically $t \simeq 0$.

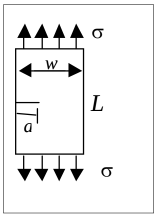
CHAPTER 4

ELASTIC STRESS FIELD EQUATIONS

4.1 (a) Calculate K_I using the singularity and non-singularity stresses for a single-edge cracked plate having a = 4 mm, x = 0.1, and $L/w \geq 1.5$, and subjected to 300 MPa in tension. **(b)** Plot the stress ratio T_x/σ and the biaxiality ratio β as functions of x = a/w.

Solution:

2) Single-edge Crack (SET)



$$K_I = \alpha\sigma\sqrt{\pi a}$$

$$L/w \geq 1.5 \text{ and } x = a/w$$

$$\alpha = 1.12 - 0.23(a/w) + 10.55(a/w)^2 - 21.71(a/w)^3 + 30.38(a/w)^4$$

Given data: $\sigma = 300\ MPa,\ \ x = 0.1,\ \ a = 4x10^{-3}\ m$

The stress ratio

$$\frac{T_x}{\sigma} = \frac{1}{(1-x)^2}\left(-0.526 + 0.641x + 0.2049x^2 - 0.755x^3 - 0.7974x^4 + 0.1966x^5\right)$$

$$\frac{T_x}{\sigma} = -0.56688$$

$$T_x = (-0.56688)\sigma = (-0.56688)(300\ MPa)$$

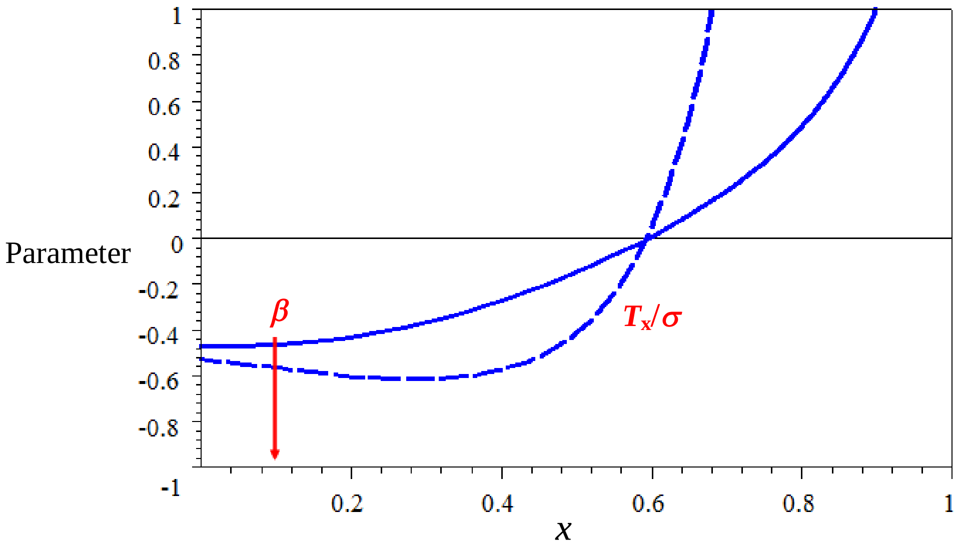
$$T_x = -170.06\ MPa$$

The biaxiality ratio

$$\beta = \frac{1}{(1-x)^{1/2}}\left(-0.469 + 0.1414x + 1.433x^2 + 0.0777x^3 - 1.6195x^4 + 0.859x^5\right)$$

$$\beta = -0.46444$$

Plots:



Stress ratio approach:

$$K_I^{(2)} = \frac{T_x}{\beta}\sqrt{\pi a} = \left(\frac{-170.06\ MPa}{-0.46444}\right)\sqrt{\pi\left(4x10^{-3}\ m\right)}$$

$$K_I^{(2)} = 41.05\ MPa\sqrt{m}$$

Classical approach:

The geometry correction factor

$$\alpha = 1.12 - 0.23(a/w) + 10.55(a/w)^2 - 21.71(a/w)^3 + 30.38(a/w)^4$$

$$\alpha = 1.12 - 0.23(0.1) + 10.55(0.1)^2 - 21.71(0.1)^3 + 30.38(0.1)^4$$

$$\alpha = 1.1838$$

The stress intensity factor

$$K_I^{(1)} = \alpha\sigma\sqrt{\pi a} = (1.1838)(300\ MPa)\sqrt{\pi\left(4x10^{-3}\ m\right)}$$

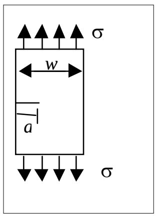
$$K_I^{(2)} = 39.81\ MPa\sqrt{m}$$

Therefore, the T_x stress approach gives slight higher values for K_I.

4.2 Using the information given in problem 4.1, calculate the stress intensity factor K_I when the stress ratio is $T_x = 0$.

Solution:

2) Single-edge Crack (SET)



$$\alpha = 1.12 - 0.23(a/w) + 10.55(a/w)^2 - 21.71(a/w)^3 + 30.38(a/w)^4$$

$a = 4x10^{-3}\ m$ $\qquad \sigma = 300\ MPa$ $\qquad x = a/w = 0.1$

$$\alpha = 1.12 - 0.23x + 10.55x^2 - 21.71x^3 + 30.38x^4$$
$$\alpha = 1.12 - 0.23(0.1) + 10.55(0.1)^2 - 21.71(0.1)^3 + 30.38(0.1)^4$$
$$\alpha = 1.1838$$
$$K_I = \alpha\sigma\sqrt{\pi a} = (1.1838)(300\ MPa)\sqrt{\pi(4x10^{-3}\ m)} =$$
$$\boxed{K_I = 39.81\, MPa\sqrt{m}}$$

4.3 Assume that a single-edge crack in a plate is loaded in tension. Derive the dominant near crack-tip stresses in Cartesian coordinates using the Westergaard complex function

$$Z(z) = \frac{K_I}{\sqrt{2\pi z}} \quad \text{where} \quad K_I = \sigma\sqrt{\pi a}$$

Solution:

Polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2},$$
$$\frac{1}{2}\sin\theta = \cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

From Euler's formulas,

$$z = re^{i\theta/2} = r(\cos\theta + i\sin\theta)$$
$$z = r^{-1/2}e^{-i\theta/2} = r^{-1/2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$
$$z = r^{-3/2}e^{-i3\theta/2} = r^{-3/2}\left(\cos\frac{3\theta}{2} - i\sin\frac{3\theta}{2}\right)$$

Find the derivative of the Westergaard complex function

$$Z(z)=\frac{K_I}{\sqrt{2\pi z}} \quad \& \quad Z'(z)=-\frac{K_I}{2\sqrt{2\pi}\,z^{3/2}}=-\frac{K_I}{2\sqrt{2\pi}}z^{-3/2}=-\frac{K_I}{2\sqrt{2\pi}}r^{-3/2}e^{-i3\theta/2}$$

Using the Euler's formula gives the Westergaard complex function in terms of trigonometric functions

$$Z(z)=\frac{K_I}{\sqrt{2\pi r e^{i\theta/2}}}=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}e^{-i\theta/2}=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)$$

$$Z'(z)=-\frac{K_I}{2\sqrt{2\pi}}r^{-3/2}e^{-i3\theta/2}=-\frac{K_I}{2\sqrt{2\pi}}r^{-3/2}\left(\cos\frac{3\theta}{2}-i\sin\frac{3\theta}{2}\right)$$

The real and imaginary parts of the trigonometric functions are

$$\operatorname{Re}Z(z)=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\cos\frac{\theta}{2} \qquad \operatorname{Im}Z(z)=-\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\sin\frac{\theta}{2}$$

$$\operatorname{Re}Z'(z)=-\frac{K_I}{\sqrt{2\pi}}r^{-3/2}\cos\frac{3\theta}{2} \qquad \operatorname{Im}Z'(z)=+\frac{K_I}{\sqrt{2\pi}}r^{-3/2}\sin\frac{3\theta}{2}$$

From eq. 4.11), the stress in the x-direction becomes

$$\sigma_x=\operatorname{Re}Z(z)-y\operatorname{Im}Z'(z)=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\cos\frac{\theta}{2}-(r\sin\theta)\left(\frac{K_I}{2\sqrt{2\pi}}r^{-3/2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_x=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\left(\cos\frac{\theta}{2}-\frac{1}{2}\sin\theta\sin\frac{3\theta}{2}\right)=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_x=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

Similarly,

$$\sigma_y=\operatorname{Re}Z(z)+y\operatorname{Im}Z'(z)=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\cos\frac{\theta}{2}+(r\sin\theta)\left(\frac{K_I}{2\sqrt{2\pi}}r^{-3/2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_y=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\left(\cos\frac{\theta}{2}+\frac{1}{2}\sin\theta\sin\frac{3\theta}{2}\right)=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_y=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

and

$$\tau_{xy}=-y\operatorname{Re}Z'(z)=-(r\sin\theta)\left(-\frac{K_I}{2\sqrt{2\pi}}r^{-3/2}\cos\frac{3\theta}{2}\right)=\frac{K_I}{\sqrt{2\pi}}r^{-1/2}\frac{1}{2}\sin\theta\cos\frac{3\theta}{2}$$

$$\tau_{xy}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

4.4 Consider a unit circle with its center at the origin and let function $F(z)=P(z)+iQ(z)$ be holomorphic inside the contour *C*. Also let the function $F(\zeta)$ take definite boundary values where $\zeta=e^{i\delta}$ gives the points on C. If the boundary condition is $F(\zeta)+\overline{F(\zeta)}=f(\delta)$, then determine the Cauchy Integral formulae and the integral for $F(z)$.

Solution:

Multiply the equation for the boundary condition by

$$\frac{1}{2\pi i}\frac{d\zeta}{\zeta-z}$$

and integrate around C to get the Cauchy Integral formulae as

$$\frac{1}{2\pi i}\int_C \frac{F(\zeta)d\zeta}{\zeta-z} + \frac{1}{2\pi i}\int_C \frac{\overline{F(\zeta)}d\zeta}{\zeta-z} = \frac{1}{2\pi i}\int_C \frac{f(\delta)d\zeta}{\zeta-z}$$

The first integral is

$$F(z) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)d\zeta}{\zeta-z}$$

The second integral becomes

$$\overline{F(0)} = a_o - ia_1$$

which are unknown so far. Then,

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(\delta)d\zeta}{\zeta-z} - a_o + ia_1$$

Let $z = 0$ so that

$$a_o = \frac{1}{2\pi i}\int_C \frac{F(\zeta)d\zeta}{\zeta} = \frac{1}{2\pi}\int_o^{2\pi} f(\delta)d\zeta$$

The remaining work to completely solve this problem can be found in Muskhelishvili's book, 1977 published edition, page 309-311.

4.5 Show that the fundamental combination of stresses can be defined in terms of Cauchy Integral formulae.

$$\sigma_x+\sigma_y = \frac{1}{\pi i}\int_C \frac{hd\zeta}{(\zeta-z)^2} + \frac{1}{\pi i}\int_C \frac{\bar{h}d\zeta}{(\zeta-z)^2} - \frac{1}{\pi i}\int_C \frac{h(\zeta)d\zeta}{\zeta^2}$$

$$\sigma_y-\sigma_x+2i\tau_{xy} = \frac{1}{\pi i}\int_C \frac{hd\zeta}{(\zeta-z)^3} + \frac{1}{\pi i}\int_C \frac{\bar{h}d\zeta}{(\zeta-z)^2} + \frac{1}{\pi i z^2}\int_C \frac{hd\zeta}{(\zeta-z)^2} - \frac{1}{\pi i z^2}\int_C \frac{hd\zeta}{\zeta^2}$$

Solution:

$$\gamma(z) = \frac{1}{2\pi i}\int_C \frac{h}{\zeta - z}d\zeta - \overline{a_1}z$$
$$\psi(z) = \int_C \frac{\overline{h(\zeta)}}{\zeta - z}d\zeta - \frac{\gamma'(z)}{z} + \frac{a_1}{z} \qquad (4.124)$$
$$a_1 + \overline{a_1} = \frac{1}{2\pi i}\int_C \frac{h(\zeta)}{\zeta^2}d\zeta$$
$$a_2 = \frac{1}{2\pi i}\int_C \frac{h}{\zeta^3}d\zeta$$

The resultant expressions are

$$\sigma_x + \sigma_y = 2\left[\gamma'(z) + \overline{\gamma'(z)}\right] \qquad (4.95a)$$
$$= \frac{1}{\pi i}\int_C \frac{h}{(\zeta - z)^2}d\zeta - 2\overline{a_1} + \frac{1}{\pi i}\int_C \frac{\bar{h}}{(\zeta - z)^2}d\zeta - 2a_1$$
$$= \frac{1}{\pi i}\int_C \frac{hd\zeta}{(\zeta - z)^2} + \frac{1}{\pi i}\int_C \frac{\bar{h}d\zeta}{(\zeta - z)^2} - \frac{1}{\pi i}\int_C \frac{h(\zeta)d\zeta}{\zeta^2}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\gamma''(z) + \psi'(z)]$$
$$= \frac{1}{\pi i}\int_C \frac{hd\zeta}{(\zeta - z)^3} + \frac{1}{\pi i}\int_C \frac{\bar{h}d\zeta}{(\zeta - z)^2} + \frac{2\gamma'(z)}{z^2} - \frac{2a_1}{z^2}$$
$$= \frac{1}{\pi i}\int_C \frac{hd\zeta}{(\zeta - z)^3} + \frac{1}{\pi i}\int_C \frac{\bar{h}d\zeta}{(\zeta - z)^2} + \frac{1}{\pi i z^2}\int_C \frac{hd\zeta}{(\zeta - z)^2} - \frac{2\overline{a_1}}{z^2} - \frac{2a_1}{z^2}$$
$$= \frac{1}{\pi i}\int_C \frac{hd\zeta}{(\zeta - z)^3} + \frac{1}{\pi i}\int_C \frac{\bar{h}d\zeta}{(\zeta - z)^2} + \frac{1}{\pi i z^2}\int_C \frac{hd\zeta}{(\zeta - z)^2}$$
$$- \frac{1}{\pi i z^2}\int_C \frac{hd\zeta}{\zeta^2}$$

4.6 Suppose that a plate containing a single unstressed crack (Figure 4,15) is deformed by the unknown stresses at infinity and assume that the complex potential, $f'(z) = \sigma - i\tau_{xy}$, represents the stress distributions across the crack contour *C*. **(a)** Determine the stress distribution on $y = 0$ outside the crack and **(b)** the upper and lower boundary functions for $|x| \geq a$.

Solution:

(a) Use eq. (4.124d) for $z = x + iy = x$ since $y = 0$.

$$\sigma_{yy} - i\tau_{xy} = -\frac{(\sigma_y^\infty + i\tau_{xy}^\infty)}{2}\left(1 - \frac{z}{\sqrt{z^2 - a^2}}\right)$$

In order to cover the stress distribution on the upper and lower planes the boundary function $\chi(z)$ has to be rewritten as $\chi(z) = \pm\sqrt{x_2 - a^2}$ for $x > a$ and $x < a$, respectively. Therefore, $|x| \geq a$ and the stress distribution becomes

$$\sigma_{yy} - i\tau_{xy} = \frac{(\sigma_y^\infty + i\tau_{xy}^\infty)}{2}\left(\frac{|x|}{\sqrt{x^2 - a^2}} - 1\right)$$

(b) The upper and lower boundary functions for $|x| \le a$ are

$$\chi^+(z) = -\chi^-(z) = i\sqrt{a^2 - x^2}$$

since $z - a = re^{i\theta}$ and $z + a = re^{i\theta}$ so that for a unit disk with $z = x + iy = x$

$$\chi^+(z) = \sqrt{(x-a)(x+a)} = \sqrt{x^2 - a^2} = i\sqrt{a^2 - x^2}$$
$$\chi^-(z) = -i\sqrt{a^2 - x^2}$$

and

$$\chi^+(z) = i\sqrt{a^2 - x^2}$$
$$-\chi^-(z) = i\sqrt{a^2 - x^2}$$

4.7 Consider an infinite plate subjected to a tensile remote stress (S) normal to the direction of a central crack. If the complex potentials for this type of crack are

$$\gamma(z) = \frac{S}{4}\left[2\sqrt{z^2 - a^2} - z\right]$$
$$\chi'(z) = \psi(z) = \frac{S}{2}\left[z - \frac{a^2}{\sqrt{z^2 - a^2}}\right]$$

then determine the crack tip stresses using the Westergaard stress function $Z(z) = 2\gamma'(z)$ when $x >> a$.

Solution:

$$\gamma'(z) = \frac{S}{4}\left(\frac{2z}{\sqrt{z^2 - a^2}} - 1\right)$$
$$Z(z) = 2\gamma'(z) = \frac{S}{2}\left(\frac{z}{\sqrt{z^2 - a^2}} - 1\right)$$
$$Z'(z) = Z' = \frac{S}{2}\left(\frac{1}{\sqrt{z^2 - a^2}} - \frac{z^2}{(z^2 - a^2)^{\frac{3}{2}}}\right)$$

The Westergaard stress functions, eq. (3.20) or (4.11), are

$$\sigma_x = \operatorname{Re} Z - y \operatorname{Im} Z'$$
$$\sigma_y = \operatorname{Re} Z + y \operatorname{Im} Z'$$
$$\tau_{xy} = -y \operatorname{Re} Z'$$

At $y = 0$ and $z = x + iy = x > a$, these stress functions become

$$\sigma_x = \operatorname{Re} Z = \frac{S}{2}\left(\frac{z}{\sqrt{z^2 - a^2}} - 1\right) = \frac{S}{2}\left(\frac{x}{\sqrt{x^2 - a^2}} - 1\right)$$

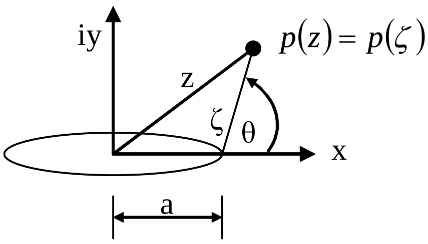
$$\sigma_y = \operatorname{Re} Z = \frac{S}{2}\left(\frac{x}{\sqrt{x^2 - a^2}} - 1\right)$$

$$\tau_{xy} = 0$$

For large $z = x + iy = x$ since $y = 0$, the elastic stresses far from the crack tip vanish when $x >> a$ along the x-axis. Therefore,

$$\sigma_x = 0$$
$$\sigma_y = 0$$
$$\tau_{xy} = 0$$

4.8 Consider the elliptical crack shown below and assume that the crack is in the z-plane where $z = \zeta + a$ and $p(z) = p(\zeta)$. Derive the stress equations using the given crack geometry and the Westergaard complex method.



Solution:

Let us move the origin to crack tip. Then the coordinates of P can be expressed in complex variable $z = \zeta + a$. Use the Westergaard complex equation so that

$$Z(z) = \frac{\sigma}{\sqrt{1 - (a/z)^2}} = \frac{\sigma z}{\sqrt{z^2 - a^2}}$$

$$Z(\zeta) = \frac{\sigma(\zeta + a)}{\sqrt{\zeta(\zeta + 2a)}} = \frac{\sigma a\left(1 + \frac{\zeta}{a}\right)}{\sqrt{2a\zeta\left(1 + \frac{\zeta}{2a}\right)}}$$

For $|\zeta| << a$,

$$Z(\zeta) = \frac{\sigma a}{\sqrt{2a\zeta}} = \frac{\sigma\sqrt{a}}{\sqrt{2\zeta}}$$
$$Z'(\zeta) = -\frac{\sigma\sqrt{a}}{(2\zeta)^{3/2}}$$

But $\zeta = re^{i\theta}$ and

$$Z(r,\theta) = -\frac{\sigma\sqrt{a}}{(2re^{i\theta})^{3/2}} = \frac{\sigma\sqrt{a}}{(2r)^{3/2}}e^{-i\theta/2}$$
$$Z(r,\theta) = \frac{\sigma\sqrt{a}}{\sqrt{2r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$
$$Z'(r,\theta) = \frac{\sigma\sqrt{a}}{(2re^{i\theta})^{3/2}} = -\frac{\sigma\sqrt{a}}{(2r)^{3/2}}e^{i3\theta/2}$$
$$Z'(r,\theta) = \frac{\sigma\sqrt{a}}{(2re^{i\theta})^{3/2}} = -\frac{\sigma\sqrt{a}}{(2r)^{3/2}}\left(\cos\frac{3\theta}{2} - i\sin\frac{3\theta}{2}\right)$$

Then,

$$\sigma_y = \operatorname{Re} Z + y \operatorname{Im} Z'$$
$$\sigma_y = \frac{\sigma\sqrt{a}}{\sqrt{2r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

and

$$\sigma_x = \frac{\sigma\sqrt{a}}{\sqrt{2r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$
$$\tau_{xy} = \frac{\sigma\sqrt{a}}{\sqrt{2r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\frac{3\theta}{2}$$

4.9 Assume that an infinite plate contains a through-central crack along the x-axis. If the plate is subjected to a remote or infinite stress loading condition, $\sigma_y = \sigma_y^\infty = S, \quad \sigma_x^\infty = \tau_{xy}^\infty = 0,$ then use the Sanford [54] modified Irwin-Sih complex potential

$$2\gamma'(z) = Z(z) - \psi^*(z)$$

Here, $\gamma'(z)$ may be defined by a Cauchy integral, $Z(z)$ is the Westergaard complex function, $\psi^*(z)$ is a complex polynomial and z is the complex variable define as $z = x + iy$. Based on this information, determine a function for the stress intensity factor K_I and expand $\psi^*(z)$ when $n = 0$ and 1, and $z = a$, half the crack length.

Solution:

The Westergaard complex function and the complex polynomial are

$$Z(z) = \frac{K_I}{\sqrt{2\pi z}}$$
$$\psi^*(z) = \sum_{n=0}^{m} b_n z^{n/2}$$

The given complex function yields

$$\frac{K_I}{\sqrt{2\pi z}} = \left[2\gamma'(z) + \sum_{n=0}^{m} b_n z^{n/2}\right]$$

For n = 0 and 1 and $z = a$,

$$K_I = \sqrt{2\pi a}\,[2\gamma'(z) + b_o + b_1 z^{1/2}]$$

CHAPTER 5

CRACK TIP PLASTICITY

5.1 Use the inequality $K_{IC} \geq K_I$ as a criterion for crack instability where K_I is defined by Irwin's plastic zone corrected expression for a finite size, to determine if a steel pressure vessel is susceptible to explode under $\sigma = 200\,MPa$ hoop stress. The vessel contains an internal circular crack perpendicular to the hoop stress. If $K_{IC} = 60\,MPa\sqrt{m}$, $\sigma_{ys} = 700\,MPa$, and the crack size is 20 mm, **(a)** determine the ASTM E399 thickness requirement and the minimum thickness to be used to prevent explosion, **(b)** Will crack propagation occur at 200 MPa? **(c)** Plot B (thickness) vs. σ/σ_{ys} for a = 10, 20, and 30 mm, and **(d)** Will the pressure vessel explode when the crack size is 30 mm? Why? or Why not? and e) When will the pressure vessel explode?

Solution:

(a) From eq. (5.12) along with $\alpha = \dfrac{2}{\pi}$,

$$K_I = \alpha\sigma\sqrt{\pi a\left[1 + 0.5(\sigma/\sigma_{ys})^2\right]}$$

Let $K_{IC} > K_I$ so that

$$\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 \geq \left(\frac{K_I}{\sigma_{ys}}\right)^2$$

$$2.5\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2_{ASTM} \geq 2.5\left(\frac{K_I}{\sigma_{ys}}\right)^2_{Actual}$$

Thus,

$$B_{ASTM} \geq 2.5\left(\frac{K_I}{\sigma_{ys}}\right)^2_{Actual}$$

$$B_{ASTM} \geq 2.5\left(\frac{\alpha\sigma\sqrt{\pi a\left[1 + 0.5(\sigma/\sigma_{ys})^2\right]}}{\sigma_{YS}}\right)^2 = 2.5\left\{\pi a\alpha^2\left(\frac{\sigma}{\sigma_{ys}}\right)^2\left[1 + \frac{1}{2}\left(\frac{\sigma}{\sigma_{ys}}\right)^2\right]\right\}$$

If the equality is used, then $B_{ASTM} = B_{\min}$ is the minimum thickness required by the ASTM E399 standard testing method. Substituting values, such as $\sigma/\sigma_{ys} = 0.285714$, a = 20 mm, and $\alpha = 2/\pi$ in the derived thickness expression, yields $B_{ASTM} = B_{\min} = 5.41\, mm$.

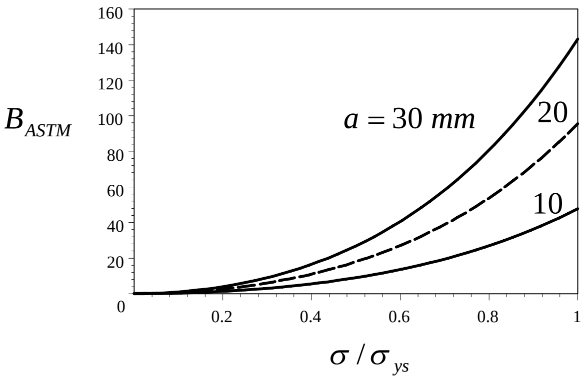
(b) $K_I = \alpha\sigma\sqrt{\pi a\left[1+0.5\left((\sigma/\sigma_{ys}\right)^2\right]} = 32.56\ MPa\sqrt{m}$. Therefore, crack propagation will not occur because $K_I < K_{IC}$. In this case, the crack is stable.

.

(c) Plot

$$B_{ASTM} \geq 2.5\left(\frac{\alpha\sigma\sqrt{\pi a\left[1+0.5(\sigma/\sigma_{ys})^2\right]}}{\sigma_{YS}}\right)^2$$

$$B_{ASTM} \geq 2.5\left\{\pi a\alpha^2\left(\frac{\sigma}{\sigma_{ys}}\right)^2\left[1+\frac{1}{2}\left(\frac{\sigma}{\sigma_{ys}}\right)^2\right]\right\}$$



(d) It will not explode since $K_I = \alpha\sigma\sqrt{\pi a_c\left[1+0.5\left((\sigma/\sigma_{ys}\right)^2\right]} \cong 40\ MPa\sqrt{m}$ and $K_I < K_{IC}$.

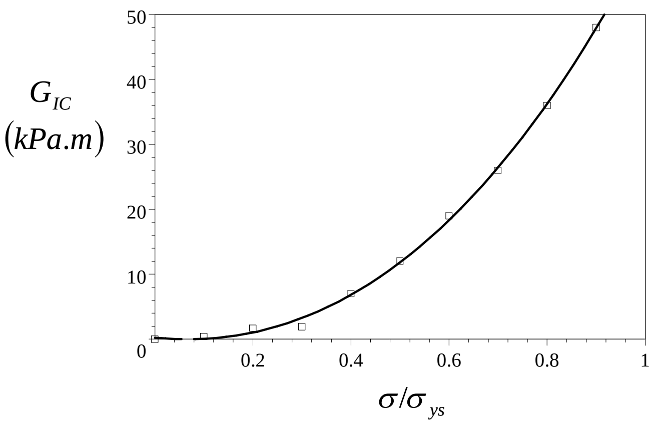
(e) Using $K_{IC} = \alpha\sigma\sqrt{\pi a_c\left[1+0.5\left((\sigma/\sigma_{ys}\right)^2\right]}$ yields the critical or maximum crack size as $a_c = 67.91\, mm$. Therefore, the pressure vessel will explode when $a = a_c = 67.91\, mm$.

5.2 A project was carried out to measure the elastic-strain energy release rate as a function of normalized stress (σ/σ_{ys}) of large plates made out a hypothetical brittle solid. All specimens had a single-edge crack of 3-mm long. Plot the given data and do regression analysis on this data set. Determine **(a)** the maximum allowable σ/σ_{ys} ratio for $G_{IC} = 30\,kPa.m$ and **(b)** K_{IC} in $MPa\sqrt{m}$. Given data: $v = 0.3$, $\sigma_{ys} = 900\,MPa$ and $E = 207\,GPa$.

σ/σ_{ys}	0	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$G_{IC}\ (kPa.m)$	0	0.40	1.70	1.90	7.00	12.00	19.00	26.00	36.00	48.00

Solution: The required regression equation and the plot are given below and the

$$G_{IC} = 0.2322 - 7.3168\left(\frac{\sigma}{\sigma_{ys}}\right) + 53.4440\left(\frac{\sigma}{\sigma_{ys}}\right)^2 + 14.9570\left(\frac{\sigma}{\sigma_{ys}}\right)^3$$



(a) Combining eqs. (3.5) and (5.12) yields

$$G_{IC} = \frac{\pi\alpha^2(1-v^2)a\sigma_{ys}(\sigma/\sigma_{ys})^2\left[1+0.5(\sigma/\sigma_{ys})^2\right]}{E}$$

$$G_{IC} = \frac{\pi(1.12)^2(1-0.3^2)(3x10^{-3}\ m)(900\ MPa)x^2\left[1+0.5x^2\right]}{207{,}000\ MPa} = 42.0980x^2 + 21.0940x^4$$

where $x = \sigma/\sigma_{ys} = 0.75$ at $G_{IC} = 30\ kPa.m.$

(b) The fracture stress and the plane strain fracture toughness

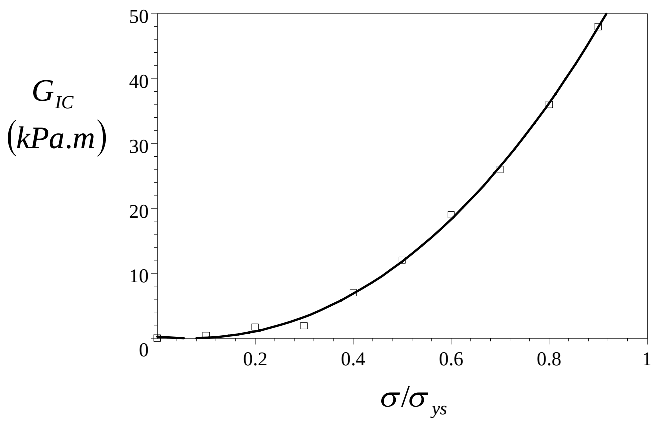
$$\sigma = (0.75)(900\ MPa) = 675\ MPa$$

$$K_{IC} = \sqrt{\frac{EG_{IC}}{1-v^2}} = \sqrt{\frac{(207{,}000\ MPa)(0.03\ MPa.m)}{1-0.3^2}} = 82.61\ MPa\sqrt{m}$$

5.3 Determine the critical crack length of problem 5.2.

Solution:

σ/σ_{ys}	0	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$G_{IC}\ (kPa.m)$	0	0.40	1.70	1.90	7.00	12.00	19.00	26.00	36.00	48.00



Combining eqs. (3.5) and (5.12) yields

$$G_{IC} = \frac{\pi\alpha^2(1-v^2)a\sigma_{ys}(\sigma/\sigma_{ys})^2\left[1+0.5(\sigma/\sigma_{ys})^2\right]}{E}$$

$$G_{IC} = \frac{\pi(1.12)^2(1-0.3^2)(3x10^{-3}\ m)(900\ MPa)x^2\left[1+0.5x^2\right]}{207{,}000\ MPa} = 42.098x^2 + 21.094x^4$$

where $x = \sigma/\sigma_{ys} = 0.75$ at $G_{IC} = 30\ kPa.m.$ The fracture stress and the plane strain fracture toughness

$$\sigma = (0.75)(900\ MPa) = 675\ MPa$$

$$K_{IC} = \sqrt{\frac{EG_{IC}}{1-v^2}} = \sqrt{\frac{(207{,}000\ MPa)(0.03\ MPa.m)}{1-0.3^2}} = 82.61\ MPa\sqrt{m}$$

Thus,

$$K_{IC} = \alpha\sigma\sqrt{\pi a_c\left[1+0.5\left(\frac{\sigma}{\sigma_{ys}}\right)^2\right]}$$

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{\alpha\sigma}\right)^2\left[1+0.5\left(\frac{\sigma}{\sigma_{ys}}\right)^2\right]^{-1} = \frac{1}{\pi}\left[\frac{82.61\ MPa\sqrt{m}}{(1.12)(675\ MPa)}\right]^2\left[1+0.5(0.75)^2\right]^{-1} = 3\ mm$$

5.4 A large brittle plate containing a central crack 4-mm long is subjected to a tensile stress of 800 MPa. The material has K_{IC} = 80 MPa$\sqrt{m}$, σ_{ys} = 1200 MPa and v = 0.30. Calculate **(a)** the applied K_I, **(b)** the plastic zone size using the Von Mises yield criterion and prove that $r = r_{max}$ when $\theta = \theta_o$. Consider all calculations for plane stress and plane strain conditions, and **(c)** draw the entire plastic zone contour where the crack tip is the origin of the coordinates.

Solution:

(a) $K_I = \alpha\sigma\sqrt{\pi a} = (800\,MPa)\sqrt{\pi\left(2\times10^{-3}\,m\right)} = 63.41\,MPa\sqrt{m}$ with $\alpha = f(a/w) = 1$

The crack is stable since $K_I < K_{IC}$.

(b) From eq. (5.52),

$$r = \frac{K_I^{\,2}}{4\pi\sigma_{ys}^{\,2}}\left[\frac{3}{2}\sin^2\theta + h(1+\cos\theta)\right]; \qquad h = \begin{Bmatrix} 1 & \text{For plane stress} \\ (1-2v)^2 & \text{For plane strain}\end{Bmatrix}$$

If θ = 0, then

$$r = \frac{hK_I^{\,2}}{2\pi\sigma_{ys}^{\,2}} \text{ and } r = \begin{Bmatrix} 0.44\,mm & \text{plane stress} \\ 0.07\,mm & \text{plane strain}\end{Bmatrix}$$

Using eq. (5.52) yields

$$\frac{\partial r}{\partial\theta} = \frac{1}{\pi}\left(\frac{K_I}{2\sigma_{ys}}\right)^2 \sin\theta[3\cos\theta - h]$$

For maximum K_I and critical plastic zone size $r = r_c$, let $\frac{\partial r}{\partial\theta} = 0$ so that

$$\sin\theta_o[3\cos\theta_o - h] = 0$$

$$\cos\theta_o = \frac{h}{3}$$

$$\theta_o = \begin{Bmatrix} 70.53^\circ & \text{For plane stress} \\ 86.94^\circ & \text{For plane strain}\end{Bmatrix}$$

Thus,

$$r_{max} = \frac{K_I^{\,2}}{4\pi\sigma_{ys}^{\,2}}\left[\frac{3}{2}\sin^2\theta_o + h(1+\cos\theta_o)\right] = \begin{Bmatrix} 0.59\ \text{mm} & \text{For plane stress} \\ 0.37\ mm & \text{For plane strain}\end{Bmatrix}$$

Furthermore,

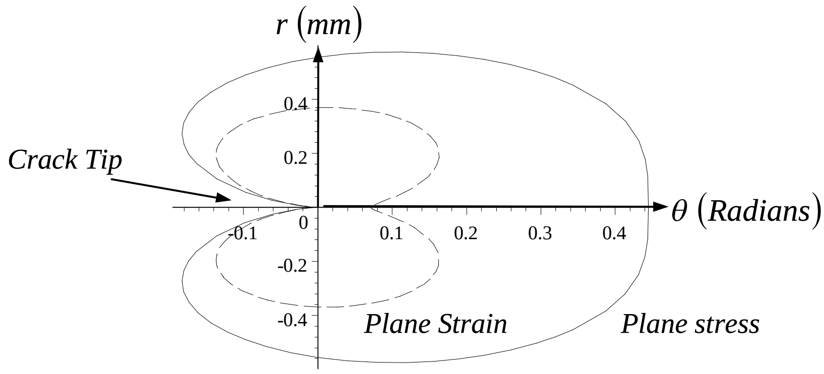
$$\frac{\partial r}{\partial\theta} = \frac{K_I}{4\pi\sigma_{ys}^{\,2}}\sin\theta[3\cos\theta - h]$$

$$\frac{\partial^2 r}{\partial \theta^2} = \frac{K_I^{\ 2}}{4\pi\sigma_{ys}^{\ 2}}\left[3\cos\theta - 3\sin^2\theta - h\cos\theta\right] = \frac{K_I^{\ 2}}{4\pi\sigma_{ys}^{\ 2}}\left[(3-h)\cos\theta - 3\sin^2\theta\right]$$

$$\left.\frac{\partial^2 r}{\partial \theta^2}\right|_{\theta_o} = \begin{Bmatrix} -0.44\ mm & \text{plane stress} \\ -0.63\ mm & \text{plane strain} \end{Bmatrix}$$

Therefore, $r = r_{max}$ at θ_o since $\frac{\partial^2 r}{\partial \theta^2} < 0$ in both cases.

(c) The contours in polar coordinates are as per $r = \frac{K_I^{\ 2}}{4\pi\sigma_{ys}^{\ 2}}\left[\frac{3}{2}\sin^2\theta + h(1+\cos\theta)\right]$



5.5 **(a)** Use the data given in Example 3.3 for a pressure vessel containing a semi-elliptical crack (Figure 3.6) to calculate Irwin's and Dugdale's **(a)** plastic zones, **(b)** K_I using Kabayashi's finite size correction factor (α) and plasticity correction factor. **(c)** Compare results and determine the percent error against each case. **(d)** Is it necessary to include a plastic correction factor? Explain.

Solution:

Data from Example 3.3:
a = 3 mm, 2c = 10 mm, B = 6 mm, $\sigma = 420\ MPa$, $\sigma_{ys} = 700\ MPa$, *a/(2c) = 0.30, a/B = 0.50,* $K_{IC} = 60\ MPa\sqrt{m}$, $\sigma/\sigma_{ys} = 0.60$, *M = 1,* $M_k = 1.12$, *and Q = 1.70* .

The correction factor and the stress intensity factor are, respectively

$$\alpha = 1.12 M_k M/\sqrt{Q} = 0.127$$
$$K_I = \alpha\sigma\sqrt{\pi a} = 5.18\ MPa\sqrt{m}$$

Comparison:

(a) Irwin's Approach:

$\alpha = 1.12 M_k M / \sqrt{Q} = 0.127$

$r = \frac{a}{2}\left(\sigma / \sigma_{ys}\right)^2 = 0.54\ mm$ Eq. (5.13)

Dugdale's Approach:

$\alpha = 1.12 M_k M / \sqrt{Q} = 0.127$

$r = \frac{a}{2}\left(\pi\sigma / 2\sigma_{ys}\right)^2 = 1.33\ mm$ Eq. (5.24)

(b) Using a finite correction factor

Irwin's:

$K_I = \alpha\sigma\sqrt{\pi(a+r)}$

$K_I = \alpha\sigma\sqrt{\pi a\left[1 + 0.5\left(\sigma / \sigma_{ys}\right)^2\right]}$ Eq. (5.12)

$K_I = \alpha\sigma\sqrt{\pi a\left[1 + 0.5\left(\sigma / \sigma_{ys}\right)^2\right]}$

$K_I(Irwin's) = 5.63\ MPa\sqrt{m}$

Dugdale's:

$K_I = \alpha\sigma\sqrt{\pi(a+r)}$

$K_I = \alpha\sigma\sqrt{\pi a\left[1 + 0.5\left(\frac{\pi\sigma}{2\sigma_{ys}}\right)^2\right]}$ Eq.(5.25)

$K_I = \alpha\sigma\sqrt{\pi a\left[1 + 0.5\left(\frac{\pi\sigma}{2\sigma_{ys}}\right)^2\right]}$

$K_I(Dugdale's) = 6.22\ MPa\sqrt{m}$

Comparisons:

$K_I(eq.\ 3.29) = 5.18\ MPa\sqrt{m} < K_I(Irwin's) < K_I(Dugdale's) = 6.22\ MPa\sqrt{m}$

Dugdale's - Irwin's $Error = 100\%(6.22 - 5.63)/5.63 \approx 10.48\%$
Irwin's - eq. (3.29) $Error = 100\%(5.63 - 5.18)/5.18 \approx 8.69\%$
Dugdale's - eq. (3.29) $Error = 100\%(6.22 - 5.18)/5.18 \approx 20.08\%$

Observe that eq. (3.29) does not include plasticity correction and it yields a smaller value than both Dugdales's and Irwin's expressions. The latter expression gives an error of 26.5%, which is slightly large. On the other hand, 8.3% error seems to be more acceptable for comparison purposes. This implies that a plastic zone correction may not be necessary.

(c) Using Irwin's approach yields a plastic zone of 0.54 mm for one side of the semi-ellipse and as a result, r << a and the plastic zone size could have been excluded. On the other hand, Dugdale's approach gives a plastic zone of 1.33 mm, which is large enough to be excluded; therefore, a plasticity correction is needed to obtain more a accurate K_I- result.

5.6 A 50- mm thick pressure vessel is to support a hoop stress of 300 MPa at room temperature under no action of corrosive agents. Assume that a semi-elliptical crack (Figure 3.6) is likely to develop on the inner surface with the major axis 2c = 40 mm and semi-minor axis a = 10 mm. A 300-M steel, which is normally used for airplane landing gear, is to be considered. Will crack propagation occur at 300 MPa hoop stress? Make sure you include the Irwin's plastic zone correction in your calculations and explain if it is necessary to do so. Use the data below and select the suitable tempered steel.

300-M Steel	σ_{ys} (MPa)	K_{IC} $(MPa\sqrt{m})$
650° Temper	1070	152
300° Temper	1740	65

Solution:

a = 10 mm, 2c = 40 mm, B = 50 mm, a/2c = 0.25, a/B = 0.20

M_k = 1.02 (From Figure 3.6b)

$\alpha = 1.12 M_k / \sqrt{Q}$ since M < 0.5 [lower limit according to eq. (3.46)]

From eq. (3.42),

$$Q = \Phi^2 - 0.212(\sigma/\sigma_{ys})^2$$

$$\Phi = \int_0^{\pi/2} \sqrt{1 - \left(\frac{c^2 - a^2}{c^2}\right)\sin^2\rho} \cdot d\rho = \int_0^{\pi/2} \sqrt{1 - 0.75\sin^2\rho} \cdot d\rho$$

$\Phi = 0.13169$ (From a Table of Elliptic Integral of the Second Kind)

Thus,

$$K_I = \alpha\sigma\sqrt{\pi a\left[1 + 0.50(\sigma/\sigma_{ys})^2\right]}$$

$$r = \frac{a}{2}(\sigma/\sigma_{ys})^2$$

<u>300-M Steel (650° C Temper)</u>

$\sigma/\sigma_{ys} = 0.28$

$Q = 1.72$

$\alpha = 0.87$

$K_I = 47.16\ MPa\sqrt{m} < K_{IC}$

r = 0.39 mm

r << a = 10 mm

<u>300-M Steel (300° C Temper)</u>

$\sigma/\sigma_{ys} = 0.17$

$Q = 1.67$

$\alpha = 0.88$

$K_I = 47.13\ MPa\sqrt{m} < K_{IC}$

r = 0.15 mm

r << a = 10 mm

These results are similar and crack propagation will not occur since both steels have $K_I < K_{IC}$. Select 300-M Steel (650° C Temper) since it has a larger K_{IC} value. The plastic zone correction was not necessary. Thus, $K_I = \alpha\sigma\sqrt{\pi a} = 46.26\ MPa\sqrt{m} < K_{IC}$. Both steels can be used for the sought application. In fact, the calculated K_I values are similar for both steels. This is purely accidental. Anyway, the plastically corrected K_I values are approximately 2% higher.

5.7 If localized plasticity is to be considered, explain the physical meaning of the following inequality $\frac{\pi a \sigma^2}{E} > \delta_t \sigma_{ys}$.

Solution:
If the applied stress (σ) is unchanged, but causes crack growth, then $\delta_t \sigma_{ys}$ is constant since $\sigma_{ys} =$ constant and $\delta_t = \delta_c$. In this case, the crack length is the only changing variable that increases; therefore, the inequality holds.

5.8 Show that $r = \delta_t /(\pi \varepsilon_{ys})$, where r is the plastic zone due to dislocation networks within the plastic zone area ahead of the crack tip.

Solution:
If $\delta_t = \frac{K_I^2}{\sigma_{ys} E}$ and $E = \frac{\sigma_{ys}}{\varepsilon_{ys}}$, then, $\delta_t = \varepsilon_{ys} \left(\frac{K_I}{\sigma_{ys}} \right)^2 = \frac{2\pi \varepsilon_{ys}}{2\pi} \left(\frac{K_I}{\sigma_{ys}} \right)^2 = 2\pi \varepsilon_{ys} \cdot r$

Thus, $r = \frac{\delta_t}{2\pi \varepsilon_{ys}}$.

5.9 Show that $\delta_t / \varepsilon_{ys} = (K_I / \sigma_{ys})^2$ and give a reasonable interpretation of this equality.

Solution: Apparently, $\delta_t / \varepsilon_{ys}$ is related to the crack size [$\delta_t / \varepsilon_{ys} \propto a$] since $(K_I / \sigma_{ys})^2$ is associated with the plastic zone [$r \propto (K_I / \sigma_{ys})^2$]. Therefore, $r \propto \delta_t / \varepsilon_{ys}$. Here $\propto$ means "proportional."

5.10 A large plate containing a through-the-thickness central crack of $2a_c = 20\ mm$ has $E = 207\ GPa$, $\sigma_{ys} = 1{,}275\ MPa$, and $\delta_c = 9.47\ \mu m$ at service temperature. Determine **(a)** the plane strain fracture toughness, **(b)** the design stress intensity factor for a safety factor (SF) of 2, **(c)** the critical fracture stress, and **(d)** the design service stress.

Solution:

(a) $\delta_c = \frac{K_{IC}^2}{\sigma_{ys} E}$

$K_{IC} = 50\ MPa\sqrt{m}$

(b) $K_{ID} = \frac{K_{IC}}{SF} = 25\ MPa\sqrt{m}$

(c) $\sigma_c = K_{IC} / \sqrt{\pi a_c} = 282\ MPa$

(d) $\sigma_d = \sigma_c / SF = 141\ MPa$

5.11 Predict δ_t for glass using $\delta = \frac{4\sigma}{E}\sqrt{a^2 - x^2}$

Solution: Glass is a brittle material and there is no need for plasticity correction; therefore, $\delta_t = 0 \quad @\ x = a.$

5.12 Develop a $\delta_t -$expression for a von Mises material. Compare it with δ_t for a Tresca material under plane strain condition. Assume that crack growth occurs along the crack plane.

Solution:

von Mises δ_t – expression:

From eq. (5.53),

$$r = \frac{hK_I^2}{2\pi\sigma_{ys}^2} \quad \text{and} \quad \delta_t = \frac{K_I\sqrt{r}}{E\sqrt{2\pi}}$$

$$\delta_t(von\ Mises) = \frac{h^{1/2}K_I^2}{2\pi E\sigma_{ys}}$$

Tresca δ_t – expression:

From eq. (5.59) with $\theta = 0$,

$$r = \frac{K_I^2}{2\pi\sigma_{ys}^2} \quad \text{and} \quad \delta_t = \frac{K_I\sqrt{r}}{E\sqrt{2\pi}}$$

$$\delta_t(Tresca) = \frac{K_I^2}{2\pi E\sigma_{ys}}$$

Combining $\delta_t(Mises)\, and\ \delta_t(Tresca)$ yields

$$\delta_t(Mises) = h^{1/2}\delta_t(Tresca)$$

$$\delta_t(Mises) = (1 - 2v)\delta_t(Tresca)$$

For Poisson's ratio $v < 1$, $\delta_t(Mises) < \delta_t(Tresca)$ because $r^{Mises} < r^{Tresca}$

5.13 A material has $E = 70\ GPa$, $\sigma_{ys} = 500\ MPa$ and $v = 1/3$. It has to be used as a plate in a large structure. Non-destructive evaluation detects a central crack of 50 mm long. If the displacement at fracture is 0.007 mm and the plate width is three times the thickness, calculate **(a)** the crack tip opening displacement, **(b)** the plane strain fracture toughness, **(c)** the plane strain energy release rate, **(d)** the plate thickness and **(e)** What's the safety factor being indirectly included in this elastic-plastic fracture mechanics approach? Assume plane strain conditions as per eq. (5.49) and a fracture load of 200 kN.

Solution:

Given data:
a = 25 mm, $\mu_f = 0.007\ mm$, $\beta = 3$ and $v = 1/3$. $\sigma_{ys} = 500\ MPa$, P = 200 kN and $E = 70\ GPa$.

(a) $\delta_{tc} = 2\mu_f = 0.014\ mm = 1.40x10^{-5}\ m$

(b) From eq. (5.49) with $\lambda = \sqrt{3} = 1.7321$, which is defined below eq. (5.5), the plane strain fracture toughness (K_{IC}), as per Irwin's model, is

$$K_{IC} = \frac{1}{2}\sqrt{\pi\lambda\delta_{tc}E\sigma_{ys}} = \frac{1}{2}\sqrt{\pi\left(\sqrt{3}\right)\left(1.40x10^{-5}\ m\right)\left(70x10^{3}\ MPa\right)\left(500\ MPa\right)} = 25.82\ MPa\sqrt{m}$$

$$G_{IC} = \frac{(1-v^2)K_{IC}^2}{E} = \frac{\left(1-1/9\right)\left(25.82\ MPa\sqrt{m}\right)^2}{70x10^3\ MPa} = 8.46\ kJ/m^2$$

(c) $\sigma_c = K_{IC}/\sqrt{\pi a} = 92.13\ MPa\sqrt{m}$ and $\sigma_c = \dfrac{P_c}{wB} = \dfrac{P_c}{3B^2}$

$$B = \sqrt{\frac{P_c}{3\sigma_c}} = \sqrt{\frac{200\ kN}{\left(3\right)\left(92.13x10^3\ kN/m^2\right)}} = 26.90\ mm \text{ and } w = 3B = 80.70\ mm \text{ (Width)}$$

(d) $SF = \sigma_{ys}/\sigma_c = 500/92.13 = 5.43$

5.14 Repeat problem 5.13 using eq. (5.41). Compare results.

Solution:

Data: a = 25 mm, $\mu_f = 0.007\ mm$, $\beta = 3$ and $v = 1/3$.

$\sigma_{ys} = 500\ MPa$, P = 200 kN, $E = 70\ GPa$ and $\delta_c = 2\mu_f = 0.014\ mm = 1.4x10^{-5}\ m$

$\kappa = 3 - 4v = 5/3$ For plane strain

(a) $\delta_c = \dfrac{(\kappa+1)(1+v)K_{IC}^2}{4\pi E\sigma_{ys}} = \dfrac{8K_{IC}^2}{9\pi E\sigma_{ys}}$

$$K_{IC} = \sqrt{\frac{9\pi\delta_c E\sigma_{ys}}{8}} = \sqrt{\frac{\left(9\pi\right)\left(1.4x10^{-5}\ m\right)\left(70x10^3\ MPa\right)\left(500\ MPa\right)}{8}} = 41.61\ MPa\sqrt{m}$$

(b) $G_{IC} = \dfrac{(1-v^2)K_{IC}^2}{E} = \dfrac{\left(1-1/9\right)\left(41.61\ MPa\sqrt{m}\right)^2}{70x10^3\ MPa} = 0.022\ MJ/m^2 = 22\ kJ/m^2$

(c) $\sigma_c = \dfrac{K_{IC}}{\sqrt{\pi a}} = \dfrac{\left(41.61\ MPa\sqrt{m}\right)}{\sqrt{\left(\pi\right)\left(25x10^{-3}\ m\right)}} = 148.48\ MPa$

$$\sigma_c = \frac{P_c}{wB} = \frac{P_c}{3B^2}$$

$$B = \sqrt{\frac{P_c}{3\sigma_c}} = \sqrt{\frac{200\ kN}{\left(3\right)\left(148.48x10^3\ kN/m^2\right)}} = 21.19\ mm \text{ and } w = 3B = 63.57\ mm \text{ (Width)}$$

(d) $SF = \dfrac{\sigma_{ys}}{\sigma_c} = 3.37$

Problem 5.13	$K_{IC} = 41.61\ MPa\sqrt{m}$	$\sigma_c = 148.48\ MPa$	$B = 21.19\ mm$	SF= 5.43
Problem 5.14	$K_{IC} = 25.82\ MPa\sqrt{m}$	$\sigma_c = 92.13\ MPa$	$B = 26.90\ mm$	$SF = 3.37$

5.15 A hypothetical large metallic plate containing a 10-mm central crack is 30-mm wide and 5-mm thick and mechanically loaded in tension. This plate has $E = 69\ GPa$, $\sigma_{ys} = 500\ MPa$, $v = 1/3$, and $\varepsilon = 0.3\%$ (plane stress strain). Determine **(a)** δ_t, **(b)** G, and c) σ as per Irwin, Dugdale, Burdekin, Rice, and Average equations, and compare the results.

Solution:

Data: $w = 30\ mm,\ B = 5\ mm,\ v = 1/3,\ a = 5\ mm,\ \varepsilon = 0.3\%,\ \sigma_{ys} = 500\ MPa,\ E = 69\ GPa$

Calculations:

$$\delta = B\varepsilon = (5\ mm)(0.003) = 0.015\ mm \quad \text{and} \quad K_I = \sigma\sqrt{\pi a}$$

$$G_I = \delta\sigma_{ys} = \left(1.5x10^{-5}\ m\right)(500\ MPa) = 0.0075\ m.MPa = 7.5 = kJ/m^2$$

(a) In general,

$$\delta_t = \frac{K_I^2}{E\sigma_{ys}} = \frac{a\pi\sigma_c^2}{E\sigma_{ys}} \quad \text{and} \quad \sigma = \sqrt{\frac{\delta_t E\sigma_{ys}}{\pi a}}$$

Irwin–eq. (5.31): $\sigma = \sqrt{\frac{\delta_t E\sigma_{ys}}{4a}} = \sqrt{\frac{(0.015\ mm)(69x10^3\ MPa)(500\ MPa)}{(4)(5\ mm)}} \cong 161\ MPa$

Dugdale–eq. (5.32): $\sigma = \sqrt{\frac{\delta_t E\sigma_{ys}}{2\pi a}} = \sqrt{\frac{(0.015\ mm)(69x10^3\ MPa)(500\ MPa)}{(2\pi)(5\ mm)}} \cong 128\ MPa$

Burdekin–eq. (5.40): $\sigma = \sqrt{\frac{\delta_t E\sigma_{ys}}{\pi a}} = \sqrt{\frac{(0.015\ mm)(69x10^3\ MPa)(500\ MPa)}{(\pi)(5\ mm)}} \cong 182\ MPa$

Rice–eq. (5.41): $\sigma = \sqrt{\frac{\delta_t E\sigma_{ys}}{a}} = \sqrt{\frac{(0.015\ mm)(69x10^3\ MPa)(500\ MPa)}{(5\ mm)}} \cong 322\ MPa$

Rice equation yields $\sigma(Rice) = 2\sigma(Irwin)$.

5.16 Determine **(a)** the critical crack tip opening displacement (δ_c), **(b)** the plastic zone size (r) and **(c)** the fracture stress (σ_c) for a large aluminum alloy plate containing a central crack of 5-mm long. Use the data given below and assume that plane strain conditions exist. Data: $K_{IC} = 25\ MPa\sqrt{m}$, $\sigma_{ys} = 500\ MPa$, and $E = 70\ GPa$.

Solution:

(a) From eq. (5.40),

$$\delta_c = \frac{K_{IC}^2}{E\sigma_{ys}} = \frac{\left(25\ MPa\sqrt{m}\right)^2}{\left(70x10^3\right)\left(500\ MPa\right)} = 0.018\ mm$$

(b) For plane strain condition,

$$r = \frac{1}{6\pi}\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = \frac{1}{6\pi}\left(\frac{25\ MPa\sqrt{m}}{500\ MPa}\right)^2 = 0.13\ mm$$

From eq. (3.24),

$$\sigma_c = \frac{K_{IC}}{\sqrt{\pi a}} = \frac{25\ MPa\sqrt{m}}{\sqrt{(\pi)\left(2.5x10^{-3}\ m\right)}} = 282\ MPa$$

5.17 Show that $\delta \approx \delta_t\sqrt{1+\left(\frac{E}{8a\sigma}\right)^2\delta_t^2}$ for plane stress conditions, where δ is the crack opening displacement (COD) and δ_t is the crack tip opening displacement (CTOD). Schematically, plot $\delta = f(\delta_t)$ for various σ and fixed a values.

Solution:

From eq. (5.24),

$$\delta_t = \frac{4\sigma}{E}\sqrt{2ar} \qquad (5.30)$$

$$2ar = \left(\frac{E\delta_t}{4\sigma}\right)^2 \qquad (a)$$

$$r = \frac{1}{2a}\left(\frac{E\delta_t}{4\sigma}\right)^2 \qquad (b)$$

$$r^2 = \frac{1}{(2a)^2}\left(\frac{E\delta_t}{4\sigma}\right)^4 \qquad (c)$$

From eq. (5.23a),

$$\delta = \frac{4\sigma}{E}\sqrt{\left(a^2+r\right)^2 - x^2} \qquad (5.29)$$

$$\delta = \frac{4\sigma}{E}\sqrt{a^2 + 2ar + r^2 - x^2} \qquad (d)$$

If $a = x$, then (e)

$$\delta = \frac{4\sigma}{E}\sqrt{2ar + r^2} \qquad (f)$$

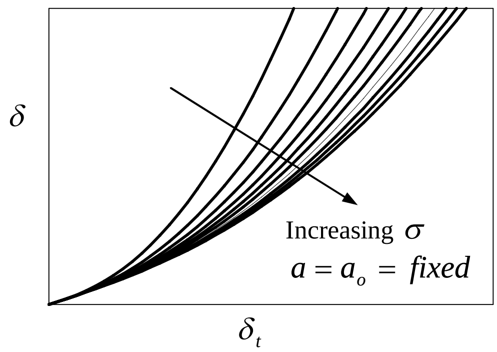
Inserting eqs. (b) and (c) into (f) yields

$$\delta = \frac{4\sigma}{E}\sqrt{\left(\frac{E}{4\sigma}\right)^2 \delta_t^2 + \left(\frac{1}{(2a)^2}\right)\left(\frac{E}{4\sigma}\right)^4 \delta_t^4}$$

$$\delta = \sqrt{\delta_t^2 + \left(\frac{1}{(2a)^2}\right)\left(\frac{E}{4\sigma}\right)^2 \delta_t^4}$$

$$\delta = \delta_t\sqrt{1 + \left(\frac{E}{8a\sigma}\right)^2 \delta_t^2}$$

The plot is



5.18 If the plane strain fracture toughness (K_{IC}) and the yield strength ($\square_{ys}$) of a 12-mm thick C(T) steel specimen are 71 MPa.m$^{1/2}$ and 1,896 MPa, respectively, determine **(a)** the validity of the fracture mechanics tension test as per ASTM E399 for the plate containing a single-edge crack of 10-mm long at fracture, **(b)** the fracture stress if the plate is 20-mm wide, **(c)** the critical crack tip opening displacement, **(d)** the plastic zone size, and **(e)** interpret the results with regard to plane strain condition. Use a Poisson's ratio of 1/3 and assume that the elastic modulus of the steel 207 GPa.

Solution:

Given data:

$K_{IC} = 71\ MPa\sqrt{m}$, $\sigma_{ys} = 1,896\ MPa$, $B = 12\ mm$

$E = 207,000\ MPa$, $v = 1/3$, $a = 10\ mm$, $w = 20\ mm$, $a/w = 0.5$

(a) Using eq. (3.5) yields the critical stress intensity factor

$$G_{IC} = \frac{K_{IC}^2}{E'} = \frac{(1-v^2)K_{IC}^2}{E} = \frac{(1-1/9)(71\ MPa\sqrt{m})^2}{207{,}000\ MPa}$$

$$G_{IC} = 2.1647 \times 10^{-2}\ MPa.m \simeq 21.65\ kPa.m$$

$$G_{IC} = 21.65\ kJ/m^2$$

The minimum size requirements can be computed using eq. (3.30). Thus,

$$a_{\min}, B_{\min} \geq 2.5\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2$$

$$a_{\min}, B_{\min} = 2.5\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = 2.5\left(\frac{71\ MPa\sqrt{m}}{1{,}896 MPa}\right)^2 = 3.51\ mm$$

Therefore, the test is valid because $a, B(10, 12) > a_{\min}, B_{\min}(3.51)\ mm$ and $a/w = 0.5$ is within the $0.2 \leq a/w \leq 1$ valid range. So proceed with the required calculations.

(b) The fracture stress is determined from eq. (3.29) along with α=9.6591 since a/w=0.5 (Consult Table 3.1).

$$\alpha = \frac{2+x}{(1-x)^{3/2}}(0.886 + 4.64x - 13.32x^2 + 14.72x^3 - 5.60x^4)$$

$$\alpha = 9.6591$$

Thus,

$$K_{IC} = \alpha\sigma_f\sqrt{\pi a}$$

$$\sigma_f = \frac{K_{IC}}{\alpha\sqrt{\pi a}} = \frac{71\ MPa\sqrt{m}}{(9.6591)\sqrt{\pi(10x10^{-3}\ m)}}$$

$$\sigma_f = 41.47\ MPa < \sigma_{ys}$$

(c) The crack-tip opening displacement is calculated using Rice's equation, eq. (5.41), with $\kappa = 3 - 4v = 5/3$ and $(\kappa + 1)(1 + v) = 32/9$.

$$\delta_{tc} = \frac{(\kappa+1)(1+v)K_{IC}^2}{4\pi E\sigma_{ys}}$$

$$\delta_{tc} = \frac{32}{36\pi}\frac{(71\ MPa\sqrt{m})^2}{(207{,}000\ MPa)(1{,}896\ MPa)} = 3.63\ \mu m$$

(d) The plastic zone can be calculated using eq. (5.53) along with the constant $h = (1 - 2v)^2 = 1/9$

$r = \frac{h}{2\pi}\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = \frac{1}{18\pi}\left(\frac{71\ MPa\sqrt{m}}{1{,}896\ MPa}\right)^2$
$r = 27.80\ \mu m$

(e) The above results suggests that the plate met the ASTM E399 size requirements because $a, B > a_{\min}, B_{\min}$ and $0.2 \le a/w \le 1$. The C(T) specimen breaks in a brittle manner because both the plastic zone size and the crack-tip opening displacement are very small.

5.19 Assume that isotropic solid material having a single-edge crack is subjected to a remote tensile stress at room temperature. Let the properties of the material be $\sigma_{ys} = 500\ MPa$, Poisson's ratio $v = 1/3$ and $E = 72\ MPa$. Let the applied stress intensity factor for mode I loading be $K_I = 20\ MPa\sqrt{m}$. Excluding microstructural details and microscale defects, use the Tresca yielding criterion to derive **(a)** an expression for the critical plastic zone angle (θ_c) and its magnitude when minimum principal stresses are equal ($\sigma_2 = \sigma_3$), **(b)** determine when $\sigma_{\min} = \sigma_2$ and $\sigma_{\min} = \sigma_3$ by knowing the value of θ_c, **(c)** the plastic zone size at θ_c and $K_I = 20\ MPa\sqrt{m}$. The Tresca yielding criterion is based on the maximum shear stress reaching a critical or failure level. Hence, the definition of the maximum shear stress for this criterion is

$$\tau_{\max} = 0.5(\sigma_{\max} - \sigma_{\min}) = 0.5\sigma_{ys}$$

Here $\sigma_{\max}$ and $\sigma_{\min}$ are principal stresses and σ_{ys} is the monotonic tensile yield strength of a solid material. Let $\sigma_{\max} = \sigma_1$ and $\sigma_{\min} = \sigma_2$ or $\sigma_{\min} = \sigma_3$.

Solution:

Needed equations for solving the problem. The principal stresses are

$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)$	5.47
$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)$	5.48
$\sigma_3 = 0$..................For plane stress	
$\sigma_3 = \frac{2vK_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$.......For plane strain	5.50

Hence,

$\sigma_1 = \sigma_{ys}$.....For plane stress	5.57
$\sigma_1 - \sigma_2 = \sigma_{ys}$.....For plane strain	5.58
$\sigma_1 - \sigma_3 = \sigma_{ys}$.....For plane strain	5.59

(a) The expression for the critical plastic zone angle (θ_c). For plane strain and $\theta_c = \theta$ conditions,

$$\sigma_2 = \sigma_3$$

$$\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right) = \frac{2vK_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$1 - \sin\frac{\theta}{2} = 2v$$

$$\theta_c = 2\arcsin(1 - 2v)$$

If $v = 1/3$, then

$$\theta_c = (2)\arcsin(1 - 2/3) = 0.67967 \text{ rad}$$

$$\theta_c = 38.942^o$$

and the minimum principal stresses become

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta_c}{2}\left(1 - \sin\frac{\theta_c}{2}\right)$$

$$\sigma_2 = \frac{(20)}{\sqrt{2\pi r}}\left[\cos\left(\frac{38.942}{2}\right)\right]\left[1 - \sin\left(\frac{38.942}{2}\right)\right] = \frac{5.015}{\sqrt{r}}$$

$$\sigma_3 = \frac{2vK_I}{\sqrt{2\pi r}} \cos\frac{\theta_c}{2} = \frac{(2/3)(20)}{\sqrt{2\pi r}} \cos\left(\frac{38.942}{2}\right) = \frac{5.015}{\sqrt{r}}$$

(b) The minimum stresses. Let's calculate the values of σ_2 and σ_3 at $\theta < \theta_c$ and $\theta > \theta_c$.

For $\theta < \theta_c = 0.67967\ rad = 38.942^o$, say $\theta = 35^o$ so that

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)$$

$$\sigma_2 = \frac{(20)}{\sqrt{2\pi r}}\left[\cos\left(\frac{35}{2}\right)\right]\left[1 - \sin\left(\frac{35}{2}\right)\right] = \frac{5.321}{\sqrt{r}}$$

$$\sigma_3 = \frac{(2/3)(20)}{\sqrt{2\pi r}} \cos\left(\frac{35}{2}\right) = \frac{5.073}{\sqrt{r}}$$

For $\theta = 20^o$,

$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)$
$\sigma_2 = \frac{(20)}{\sqrt{2\pi r}}\left[\cos\left(\frac{20}{2}\right)\right]\left[1 - \sin\left(\frac{20}{2}\right)\right] = \frac{6.493}{\sqrt{r}}$
$\sigma_3 = \frac{2vK_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$
$\sigma_3 = \frac{(2/3)(20)}{\sqrt{2\pi r}} \cos\left(\frac{20}{2}\right) = \frac{5.238}{\sqrt{r}}$

Therefore, the calculated values of the principal stress σ_2 and σ_3 at $\theta < \theta_c$ and fixed $\sqrt{r}$ indicate that $\sigma_{\min} = \sigma_3$.

For $\theta > \theta_c = 0.67967\ rad = 38.942^o$, say $\theta = 40^o$ so that

$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)$
$\sigma_2 = \frac{(20)}{\sqrt{2\pi r}}\left[\cos\left(\frac{40}{2}\right)\right]\left[1 - \sin\left(\frac{40}{2}\right)\right] = \frac{4.933}{\sqrt{r}}$
$\sigma_3 = \frac{2vK_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$
$\sigma_3 = \frac{(2/3)(20)}{\sqrt{2\pi r}} \cos\left(\frac{40}{2}\right) = \frac{4.998}{\sqrt{r}}$

For $\theta = 60^o$,

$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)$
$\sigma_2 = \frac{(20)}{\sqrt{2\pi r}}\left[\cos\left(\frac{60}{2}\right)\right]\left[1 - \sin\left(\frac{60}{2}\right)\right] = \frac{3.455}{\sqrt{r}}$
$\sigma_3 = \frac{2vK_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$
$\sigma_3 = \frac{(2/3)(20)}{\sqrt{2\pi r}} \cos\left(\frac{60}{2}\right) = \frac{4.607}{\sqrt{r}}$

Therefore, the calculated values of the principal stress σ_2 and σ_3 at $\theta > \theta_c$ and fixed $\sqrt{r}$ indicate that $\sigma_{\min} = \sigma_2$.

(c) The plastic zone sizes for plane stress and plane strain at $\theta_c = 38.942^o$ are

$$r(\text{eq. }5.60) = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2\left[\left(\cos\frac{\theta_c}{2}\right)\left(1+\sin\frac{\theta}{2}\right)\right]^2$$

$$r(\text{eq. }5.60) = \frac{1}{2\pi}\left(\frac{20}{500}\right)^2\left\{\cos\left(\frac{38.942}{2}\right)\left[1+\sin\left(\frac{38.942}{2}\right)\right]\right\}^2$$

$$r(\text{eq. }5.560) = 4.02\times10^{-4}\ m \simeq 0.40\ mm\ \ (\text{Plane stress at }\theta_c)$$

$$r(\text{eq. }5.61) = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2\left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^2$$

$$r(\text{eq. }5.61) = \frac{1}{2\pi}\left(\frac{20}{500}\right)^2\left(2\cos\left(\frac{38.942}{2}\right)\sin\left(\frac{38.942}{2}\right)\right)^2$$

$$r(\text{eq. }5.61) = 1.00\times10^{-4}\ m \simeq 0.10\ mm\ \ (\text{Plane strain at }\theta_c)$$

$$r(\text{eq. }5.62) = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2\cos^2\frac{\theta}{2}\left(1-2v+\sin\frac{\theta}{2}\right)^2$$

$$r(\text{eq. }5.62) = \frac{1}{2\pi}\left(\frac{20}{500}\right)^2\left[\cos^2\left(\frac{38.942}{2}\right)\right]\left[1-2/3+\sin\left(\frac{38.942}{2}\right)\right]^2$$

$$r(\text{eq. }5.62) = 1.00\times10^{-4}\ m \simeq 0.10\ mm\ \ (\text{Plane strain at }\theta_c)$$

As expected, *r* for plane stress is greater than that for plane strain.

5.20 Consider a ductile steel plate containing a 50-mm through-thickness central crack subjected to a remote tensile stress of 40 MPa. If the yield strength of the steel is 300 MPa, then calculate K_I as per LEFM, Irwin's approximation and the Dugdale's strip yield criterion. Compare results. Repeat all calculations for 290 MPa remote stress. Explain.

Solution:

Given data: $\sigma = 40\ MPa$ and $\sigma_{ys} = 300\ MPa$

LEFM: $$K_I = \sigma\sqrt{\pi a} = (40\ MPa)\sqrt{\pi\left(\frac{50}{2}\right)(10^{-3}\ m)} = 11.21\,MPa\sqrt{m}$$

Irwin's: $$K_I = \frac{\sigma\sqrt{\pi a}}{\sqrt{1-0.5(\sigma/\sigma_{ys})^2}} = \frac{(40\ MPa)\sqrt{\pi\left(\frac{50}{2}\right)(10^{-3}\ m)}}{\sqrt{1-0.5(40/300)^2}} = 11.26\ MPa\sqrt{m}$$

Dugdale's: $$K_I = \frac{\sigma\sqrt{\pi a}}{\sqrt{\frac{8}{\pi^2}\ln\left[\sec\left(\frac{\pi\sigma}{2\sigma_{ys}}\right)\right]}} = \frac{(40\ MPa)\sqrt{\pi\left(\frac{50}{2}\right)(10^{-3}\ m)}}{\sqrt{\frac{8}{\pi^2}\ln\left[\sec\left(\frac{\pi x 40}{2x300}\right)\right]}} = 11.25\,MPa\sqrt{m}$$

These results are similar. However, if $\sigma = 290\,MPa,$ then K_I

LEFM: $K_I = 81.27\ MPa\sqrt{m}$

Irwin's: $K_I = 115.18\ MPa\sqrt{m}$

Dugdale's: $K_I = 130.01\,MPa\sqrt{m}$

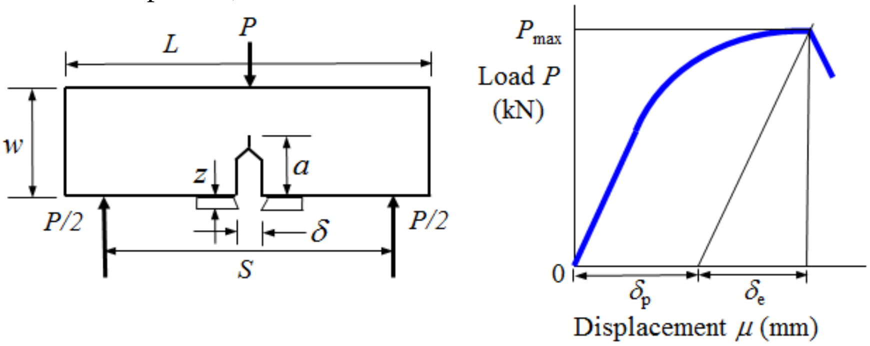
Therefore, K_I values are dubious. See Figure 5.4 for comparing Irwin's and Dugdale's approximation schemes.

5.21 A single-edge SE(B) specimen with B = 20 mm, w = 40 mm is used to determine the critical CTOD δ_{tc} and J_{IC}. See Example 5.4 for the specimen configuration and the load-displacement plot. Assume plane strain conditions and let

$$P_{\max} = 38\ kN,\ \sigma_{ys} = 800\ MPa,\ \ E = 207\ GPa,\ v = 0.3,\ K_{IC} = 60\ MPa\sqrt{m}$$
$$B = 20\ mm;\quad w = 40\ mm;\quad \delta_p = 1.00\ mm;\ S = 150\ mm;\quad z = 1.5\ mm;\quad a = 12\ mm$$

Solution:

From Example 5.4,



The geometry correction factor α :
$$x = a/w = 12/40 = 0.3$$
$$\alpha = \frac{3x^{1/2}\left\{1.99 - x(1-x)\left[2.15 - 3.93x + 2.7x^2\right]\right\}}{2(1+2x)(1-x)^{3/2}}$$
$$\alpha = \frac{3(0.3)^{1/2}\left\{1.99 - (0.3)(1-0.3)\left[2.15 - 3.93(0.3) + 2.7(0.3)^2\right]\right\}}{2(1+2*0.3)(1-0.3)^{3/2}} = 1.5212$$

The applied stress intensity factor as per LEFM is
$$K_I = \frac{\alpha P_{\max} S}{Bw^{3/2}} = \frac{(1.5212)(38000\ N)(150x10^{-3}\ m)}{(20x10^{-3}\ m)(40x10^{-3}\ m)^{3/2}} = 54.19\ MPa\sqrt{m}$$

Then, CTOD becomes
$$\delta_{te} = \frac{K_I^2}{m\sigma_{ys}E} = \frac{\left(54.19\ MPa\sqrt{m}\right)^2}{(2)(800\ MPa)(207000\ MPa)} = 8.866x10^{-3}\ m \approx 0.009\ mm$$
$$\delta_{tp} = \frac{0.44(w-a)\delta_p}{0.44(w-a)+a+z} = \frac{(0.44)(40\ mm - 12\ mm)(1\ mm)}{(0.44)(40\ mm - 12\ mm) + 12\ mm + 1.5\ mm} = 0.47715\ mm$$
$$\delta_{tc} = \delta_{te} + \delta_{tp} = 0.008866 + 0.47715 \approx 4.78\ mm$$

Thus, the J-integral is
$$J_{IC} = m\sigma_{ys}\delta_{tc} = (2)(800\ MPa)(4.78x10^{-3}\ m) = 7.65\ MPa.m = 7.65 MN/m^2$$

CHAPTER 6

THE ENERGY PRINCIPLE

6.1 Use Dugdale's model for a fully developed plane stress yielding confined to a narrow plastic zone. Yielding is localized to a narrow size roughly equal to the sheet thickness (B). This is a fully elastic case, in which the plastic strain may be defines as $\varepsilon = \delta_t/B$, where δ_t is the crack tip opening displacement. If the J-integral is defined by $dJ = \sigma d\delta$, then show that

$$\delta_t = \frac{\pi a \alpha^2 \sigma^2}{E\sigma_{ys}}$$

Solution:

If the strain energy density and the J-integral at yielding are defined by

$$W = \int_0^\varepsilon \sigma d\varepsilon = \int_0^\varepsilon \sigma_{ys} d\varepsilon$$

$$J = BW = B\int_o^\varepsilon \sigma_{ys} d\varepsilon = \delta_t \sigma_{ys}$$

Thus,

$$J = B\int_o^\varepsilon \sigma_{ys} d\varepsilon = \int_o^{\delta_t} \sigma_{ys} d\delta$$

Solving the J-integral yields

$$J = \int_o^{\delta_t} \sigma_{ys} d\delta = \delta_t \sigma_{ys}$$

Then,

$$J = \frac{K_I^2}{E} \qquad and \qquad \delta_t \sigma_{ys} = \frac{K_I^2}{E}$$

$$\delta_t = \frac{K_I^2}{E\sigma_{ys}} = \frac{\pi a \alpha^2 \sigma^2}{E\sigma_{ys}}$$

Here, α is a correction factor.

6.2 The crack tip opening displacement (δ_t) for perfectly plastic solution to the Dugdale model was derived by Rice in 1966 [21] as

$$\delta_t = \frac{(k+1)a\sigma_0}{\pi G}\log\left[\sec\left(\frac{\pi\sigma}{2\sigma_0}\right)\right]$$

where $\sigma_o = \sigma_{ys}$ and $\sigma =$ Remote tensile stress ($\sigma > \sigma_o$), $a =$ Crack size, and G = Shear modulus. Show that the path-independent J-integral is defined by

$$J = \frac{(k+1)(1+v)\pi a\sigma^2}{E} = f(a,\sigma)$$

Solution:

Using Taylor's series on $\sec\left(\frac{\pi\sigma}{2\sigma_0}\right)$ and on $\log\left[\sec\left(\frac{\pi\sigma}{2\sigma_0}\right)\right]$ yields

$$y = \frac{\pi\sigma}{2\sigma_0} \qquad \text{and} \qquad \sec(y) = 1 + \frac{y^2}{2} + \frac{5}{24}y^4 + \ldots \approx 1 + \frac{y^2}{2}$$

Let $x = \sec(y)$ so that

$$\log(x) = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \ldots\right] \qquad \text{For } x > 0$$

$$\log(x) \approx 2\left(\frac{x-1}{x+1}\right) = 2\left[\frac{\sec(y)-1}{\sec(y)+1}\right] = 2\left[\frac{1+\frac{y^2}{2}-1}{1+\frac{y^2}{2}+1}\right] = 2\left[\frac{y^2}{4+y^2}\right] = 2\left[\frac{\left(\frac{\pi\sigma}{2\sigma_o}\right)^2}{4+\left(\frac{\pi\sigma}{2\sigma_o}\right)^2}\right]$$

If $\frac{\sigma}{\sigma_o} \le 1$, then $4 >> \left(\frac{\pi\sigma}{2\sigma_o}\right)^2$ and $\log\left[\sec\left(\frac{\pi\sigma}{2\sigma_o}\right)\right] \approx \frac{1}{2}\left(\frac{\pi\sigma}{2\sigma_o}\right)^2$

Thus, $\delta_t \approx \frac{(k+1)a\sigma_0}{2\pi G}\left(\frac{\pi\sigma}{2\sigma_o}\right)^2$

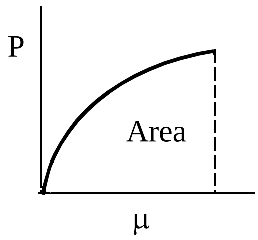
From eq. (6.66),

$$J = \delta_t\sigma_0 = \frac{(k+1)a\sigma_o^2}{2\pi G}\left(\frac{\pi\sigma}{2\sigma_o}\right)^2 = \frac{(k+1)a\pi\sigma^2}{8G}$$

The elastic shear modulus of elasticity is $G = \frac{E}{2(1+v)}$. Thus, J becomes

$$J = \frac{(k+1)(1+v)\pi a\sigma^2}{4E} = f(a,\sigma)$$

6.3 A bending test specimen made out of carbon steel showed a load-displacement behavior



A = Area =10 Joules
E = 207 GPa
σ_{ys} = 1,500 MPa
B = 25 mm
b = 25 mm

If the area under the P *vs.* μ curve is 10 joules at the onset of crack growth, determine **(a)** the fracture toughness in terms of J_{IC} as per ASTM E-813 Standard, **(b)** K_{IC} and its validity as per ASTM E399 testing method, and **(c)** δ_{tc} [according to eq. (5.31) with $\lambda = \sqrt{3}$], ε_f and r. Explain the meaning of the results.

Solution:

(a) $J_{IC} = \dfrac{2A}{Bb} = \dfrac{(2)(10\, joules)}{(25\times 10^{-3} m)(25\times 10^{-3} m)} = 32\ kJ/m^2 = 32x10^{-3}\ m.MPa$

(b) $K_{IC} = \sqrt{EJ_{IC}} = \sqrt{(207x10^3\ MPa)(32x10^{-3}\ m.MPa)} = 81.39\ MPa\sqrt{m}$

Validity: $B_{ASTM} \geq 2.5(K_{IC}/\sigma_{ys})^2 = 7.36\ mm = B_{\min}$ and $B_{Actual} > B_{ASTM}$

Therefore, $K_{IC} = 81.39\ MPa\sqrt{m}$ is valid.

(c) From eq. (5.31),

$$\delta_{tc} = \frac{4K_{IC}^2}{\pi\lambda E\sigma_{ys}} = \frac{(4)(81.39\ MPa\sqrt{m})^2}{(\pi)(\sqrt{3})(207x10^3\ MPa)(1{,}500\ MPa)} = 0.016\ mm$$

Using eqs. (5.44) and (5.13) yields the fracture strain and the plastic zone size, respectively

$$\varepsilon_f = \frac{\delta_c}{B} = \frac{0.016\ mm}{25\ mm} = 6.4x10^{-4}\ or\ 0.064\%$$

The plastic zone size:

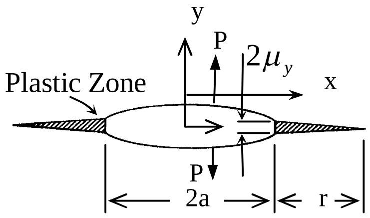
$$r = \frac{1}{6\pi}\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = \frac{1}{6\pi}\left(\frac{81.39\ MPa\sqrt{m}}{1{,}500\ MPa}\right)^2 \approx 0.16\ mm << a$$

Therefore, the material behaved in a brittle manner implying that very little plastic deformation took place at the crack tip. These are very small values indicating that the material is brittle and undergoes very little plastic deformation. With regard to the critical crack tip displacement of 0.016 mm, it indicates that the crack tip displacement is very small; that is, $\mu_y = \delta_c/2 = 0.008\ mm$.

6.4 If $J_I = \sigma_{ys}\delta_t$ is used to determine the fracture toughness, will δ_t be a path-independent entity? Explain.

Solution:

Using Dugdale's model, Figure 5.3 and eq. (6.33) yields



$$J_I = \int_a^{a+r} \left(Wdy - \vec{T}\frac{\partial \vec{\mu}}{\partial x} ds \right)$$

According to Figure 5.3, $dy = 0$ from a to $a + r$ and $\vec{T} = \sigma_y = \sigma_{ys}$. Thus,

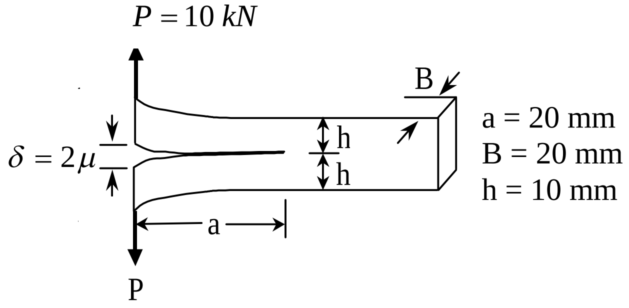
$$J_I = -\int_a^{a+r} \sigma_{ys} \frac{\partial\left(\mu_y^+ - \mu_y^-\right)}{\partial x} dx = \int_o^{\delta_t} \sigma_{ys} d\left(\mu_y^+ - \mu_y^-\right) dx = \delta_t \sigma_{ys}$$

Therefore, both J_I and δ_t are path-independent. Only the initial and final values of δ_t are needed to solve the integral. In fact, δ_t is just an opening as well as a path-independent entity.

6.5 Assume that crack growth occurs when $J_I \le J_{IC}$, where $J_{IC} = (1-\nu^2)K_{IC}^2 / E$. If a well-developed plastic flow occurs, will these expressions be valid? Explain.

Solution: It is valid in the elastic regime where $a >> r$.

6.6 A double cantilever beam (DCB) is slowly loaded in tension up 10 MN as shown schematically below. Assume that there is no rotation at the end of the beam and that the beam is made of isotropic steel having the following properties: $K_{IC} = 47\ \text{MPa}\sqrt{\text{m}}$ and $E = 207\ GPa$. Will fracture occur?



Solution:

From eq. (6.42) along with $E' = E/(1-v^2)$ and $I = Bh^3/12$,

$$G_I = \frac{P^2a^2}{BE'I} = \frac{12(1-v^2)P^2a^2}{EB^2h^3} = \frac{12(1-0.3^2)(10x10^{-3}\ MN)^2(20x10^{-3}\ m)^2}{(207{,}000\ MN/m^2)(20x10^{-3}\ m)^2(10x10^{-3}\ m)^3}$$

$$G_I = 5.28x10^{-3}\ MPa.m$$

$$K_I = \sqrt{\frac{EG_I}{1-v^2}} = \sqrt{\frac{(207{,}000\ MPa)(5.28x10^{-3}\ MPa.m)}{1-0.3^2}}$$

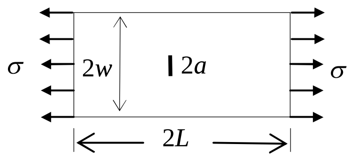
$$K_I = 34.66\ MPa\sqrt{m}$$

Therefore, fracture will not occur because $K_I = 34.66\ MPa\sqrt{m} < K_{IC} = 47\ MPa\sqrt{m}$

CHAPTER 7

PLASTIC FRACTURE MECHANICS

7.1 Determine **(a)** the J-integral (J) and **(b)** the dJ/da for a hypothetical steel plate containing a central crack of 114 mm long loaded at 276 MPa and exposed to room temperature air. What does $T_J < T_R$ mean? Assume plane strain conditions and that the stainless steel obeys the Ramberg-Osgood relation with curve fitting parameters such as $\alpha' = 1.69$ and $n = 5.42$. Explain the results based on the strain hardening effect. Data: $E = 206{,}850\ MPa$, $\sigma_{ys} = 207\ MPa$, $v = 0.3$ and $J_{IC} = 130\ kJ/m^2$. Dimensions: $w = 4a = 228\ mm$, $L = 2w$



Solution:

(a) For a central crack, use Table 3.1 for α

$$x = a/w = 57/228 = 0.25$$
$$\alpha = \sqrt{1 + 0.5x^2 + 20.46x^4 + 81.72x^6} = \sqrt{1 + 0.5(0.25)^2 + 20.46(0.25)^4 + 81.72(0.25)^6} = 1.0635$$
$$\alpha = \sqrt{\sec[\pi(0.25)]} = 1.1892$$
$$K_I = \alpha\sigma\sqrt{\pi a_e}$$

The effective crack length needed for calculating $K_I = K_I(a_e)$ is given below.

$$r_y = \left(\frac{1}{\pi\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{K_I}{\sigma_o}\right)^2 = \left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\alpha\sigma}{\sigma_o}\right)^2 a_e \qquad \& \quad \varpi = \frac{1}{1+(P/P_o)^2} \qquad \beta = 6$$

$$a_e = a + \overline{w} r_y = a + \left[\frac{1}{1+(P/P_o)^2}\right]\left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\alpha\sigma}{\sigma_o}\right)^2 a_e$$

With $\alpha = \sqrt{\sec[\pi(0.25)]} = 1.1892$

$$a_e = 57\ mm + \left[\frac{1}{1+(125.86/81.75)^2}\right]\left(\frac{1}{6}\right)\left(\frac{5.42-1}{5.42+1}\right)\left[\frac{(1.1892)(276)}{207}\right]^2 a_e$$
$$a_e = 57\ mm + 8.5597x10^{-2} a_e$$
$$a_e = 62.336\ mm$$

$$K_I = \alpha\sigma\sqrt{\pi a_e} = (1.1892)(276\ MPa)\sqrt{\pi(62.336x10^{-3}\ m)}$$
$$K_I = 145.25\ \text{MPa}\sqrt{\text{m}}$$

The elastic J-integral values that include the effects strain hardening is

$$J_e = \frac{K_I^2}{E'} = \frac{(1-\nu^2)K_I^2}{E} = \frac{(1-0.3^2)(145.25)^2}{206{,}850} = 9.2815x10^{-2}\ MPa\cdot m$$
$$J_e = 92.82\ kPa.m = 92.82\ kJ/m^2$$

Plastic Part (continued): Table 7.2 for a central crack

$$\varepsilon_o = \sigma_o/E = 0.001 \quad \text{and} \quad \varepsilon = 0.001$$
$$P_o = 4(w-a)\sigma_o/\sqrt{3} = 81.75\ MN/m \quad \& \quad P = 2w\sigma = 2(228x10^{-3}\ m)(276\ MPa) = 125.86\ MN/m$$
$$w - a = 228 - 57 = 171\ mm$$

The Plastic zone:

$$r = r_y = \left(\frac{1}{6}\right)\left(\frac{5.42-1}{5.42+1}\right)\left(\frac{145.25}{207}\right)^2(10^3) = 17.855\ mm$$

From eq. (7.25) and Table 7.1 with n = 5.42,

$$I_n = 6.2511 - 0.29381n + 1.1724x10^{-2}n^2 = 5.0031$$
$$J_p = \alpha'\sigma_o\varepsilon_o I_n r\left(\frac{\sigma}{\sigma_o}\right)^{n+1} = (1.69)(207\ MPa)(0.001)(5.0031)(17.855x10^{-3}\ m)\left(\frac{276}{207}\right)^{5.42+1}(10^3)$$
$$J_p = 87.17\ kJ/m^2$$

Thus, eq. (7.36) gives

$$\text{J} = \text{J}_\text{e}(\text{a}_\text{e}) + \text{J}_\text{p}(\text{a},\text{n}) = 92.82\ \text{kJ/m}^2 + 87.17\ \text{kJ/m}^2$$
$$J = J_I = 180\ kJ/m^2$$

Therefore, crack grows occurs since $J = 290.96\ kJ/m^2 > J_{IC} = 130\ kJ/m^2$

(b) Crack instability

$$K_I^2 = \alpha\pi(\alpha\sigma)^2$$
$$J_e = \frac{K_I^2}{E'} = \frac{(1-\nu^2)K_I^2}{E} = \frac{\pi(1-\nu^2)(\alpha\sigma)a}{E}$$

$$r = r_y = \left(\frac{1}{\pi\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{K_I}{\sigma_o}\right)^2 = \left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\alpha\sigma}{\sigma_o}\right)^2 a$$

$$J_p = \alpha' \sigma_o \varepsilon_o I_n r \left(\frac{\sigma}{\sigma_o}\right)^{n+1} = \alpha' \sigma_o \varepsilon_o I_n \left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\alpha\sigma}{\sigma_o}\right)^2 \left(\frac{\sigma}{\sigma_o}\right)^{n+1} a$$

$$J = J_e + J_p = \frac{\pi(1-\nu^2)(\alpha\sigma)a}{E} + \alpha' \sigma_o \varepsilon_o I_n \alpha^2 \left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_o}\right)^{n+3} a$$

$$\frac{dJ}{da} = \frac{\pi(1-\nu^2)(\alpha\sigma)^2}{E} + \alpha' \sigma_o \varepsilon_o I_n \alpha^2 \left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_o}\right)^{n+3} > 0$$

$$\frac{dJ}{da} = \frac{\pi(1-0.3^2)(1.1892x276\ MPa)^2}{206{,}850\ MPa}$$

$$+1.69(207\ MPa)(0.001)(5.0031)(1.0635)^2 \left(\frac{1}{6}\right)\left(\frac{5.42-1}{5.42+1}\right)\left(\frac{276}{207}\right)^{5.42+3}$$

$$\frac{dJ}{da} = 4.05\ MPa = 4.05\ MJ/m^3$$

From eq. (6.70), the tearing modulus is

$$T_J = \frac{E}{\sigma_o^2}\frac{dJ}{da} = \frac{(206{,}850\ MPa)}{(207\ MPa)^2}(4.05\ MPa) = 19.55$$

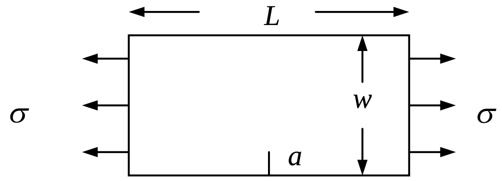
$$T_J = 19.55$$

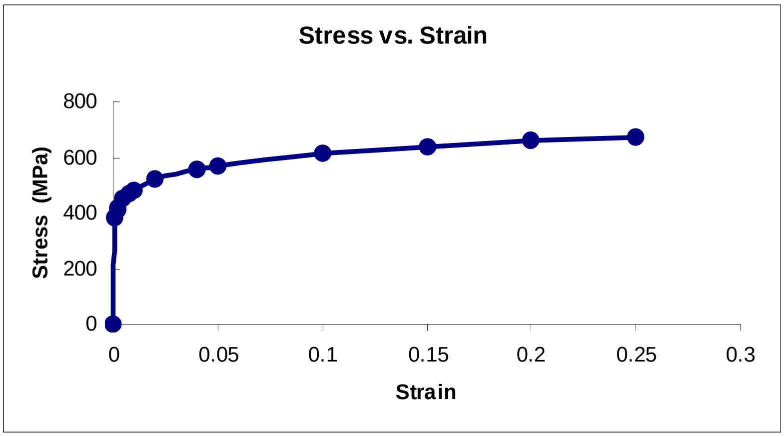
Therefore, crack growth is stable if $T_J < T_R$.

7.2 Plot the given uniaxial stress-strain data for an ASTM A533B steel at 93ºC, and perform a regression analysis based on the Ramberg-Osgood equation. A single-edge-cracked plate made out of this steel is loaded in tension at 93ºC. Determine the total elastic-plastic J-integral when the applied load is $\sigma = 500\ MPa$. Will the crack grow in a stable manner? Why? or Why not? Assume plane strain conditions and necessary assumptions. Let $a = 100\ mm$, v = 0.30, E = 207 GPa, $L = 3w$, w = 400 mm, J_{IC} = 1.20 MPa.m, $h_1 = 0.523$ and B = 150 mm.

Strain ($x10^{-3}$)	0	1.00	2.24	2.30	2.35	5.00	7.50	10	20	40
Stress (MPa)	0	381	414	415	416	450	469	483	519	557

Solution: Single-edge cracked specimen





Data for plane strain condition:

$\sigma_{ys} = \sigma_o = 414\ MPa$ (From the above Figure) $\quad a = 100\ mm \quad B = 150\ mm$

$E = 207\ GPa \quad v = 0.30 \quad w = 400\ mm$

$J_{IC} = 1.20\ MPa \cdot m \cong 6.8\ \dfrac{in \cdot kips}{in^2}$ (From Ref. [8]) $\quad L = 3w = 1{,}200\ mm$

$\sigma = 500\ MPa$ (Applied) $\quad b = w - a = 300\ mm$ (Ligament)

Elastic Part:

Nonlinear regression using $\varepsilon = \alpha' \varepsilon_0 \left(\frac{\sigma}{\sigma_0} \right)^n$ yields $\alpha' = 1.12$ and $n = 9.71$. From Table 7.2,

$$P = w\sigma = \left(400x10^{-3}\ m\right)\left(500\ MPa\right) = 200\ MN/m$$

$$x = \left[1 + \left(\frac{a}{w-a}\right)^2\right]^{1/2} - \frac{a}{w-a} = \left[1 + \left(\frac{100}{400-100}\right)^2\right]^{1/2} - \frac{100}{400-a}100 = 0.82137$$
$$P_0 = 1.455x(w-a)\sigma_0 = (1.455)(0.82137)\left(300x10^{-3}\ m\right)\left(414\ MPa\right) = 148.43\ MN/m$$

$$\lambda_1 = (w-a)(a/w) = (400-100)(100/400) = 75\ mm$$

Also, $\beta = 6$ and

$$r_y = \left(\frac{1}{\pi\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{K_I}{\sigma_0}\right)^2 = \left(\frac{1}{\pi\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma\sqrt{\pi a_e}}{\sigma_0}\right)^2 = \left(\frac{a_e}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_0}\right)^2$$
$$\overline{w} = \frac{1}{1 + \left(P/P_0\right)^2}$$

The crack size can be calculated using

$$a_e = a + \overline{w} r_y$$
$$a_e = a + \left[\frac{1}{1 + \left(P/P_0\right)^2}\right]\left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_0}\right)^2 a_e = a + \left[\frac{1}{1 + (200/148.43)^2}\right]\left(\frac{1}{6}\right)\left(\frac{9.71-1}{9.71+1}\right)\left(\frac{500}{414}\right)(a_e)$$
$$a_e = a + 0.05814 a_e$$
$$a_e = 1.0617a$$

$$a = 100\ mm$$
$$a_e = 1.0617(100\ mm) = 106.17\ mm$$

For $a/w = 100/400 = 0.25$,
$$\alpha = 1.12 - 0.23(a/w) + 10.55(a/w)^2 - 21.71(a/w)^3 + 30.38(a/w)^4 = 1.5013$$

$$K_I = \alpha\sigma\sqrt{\pi a_e} = (1.5013)(500\ MPa)\sqrt{(\pi)\left(106.17x10^{-3}\ m\right)} = 469.10\ MPa\sqrt{m}$$
$$J_e = \frac{K_I^2}{E'} = \frac{\left(1-\nu^2\right)K_I^2}{E} = \frac{\left(1-0.3^2\right)\left(469.10\ MPa\sqrt{m}\right)^2}{207x10^3\ MPa} = 0.97\ m.MPa = 0.97\ MJ/m^2$$

Plastic part:

$$r = r_y = \left(\frac{a_e}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_0}\right)^2 = \left(\frac{106.17\,mm}{6}\right)\left(\frac{9.71-1}{9.71+1}\right)\left(\frac{500}{414}\right)^2 = 21\,mm$$

From eq. (7.51) and Table 7.2,
$h_1 = 0.523$, $\alpha' = 1.12$ and $\lambda_1 = 75\,mm = 75x10^{-3}\,m$

$$J_p = \alpha'\sigma_0\varepsilon_0\lambda_1 h_1\left(\frac{P}{P_0}\right)^{n+1} \quad with \quad \varepsilon_0 = \sigma_0/E = 0.002$$

$$J_p = (1.12)(414\,MPa)(0.002)(75x10^{-3}\,m)(0.523)\left(\frac{200}{148.43}\right)^{1+9.71}$$

$$J_p = 0.89\,MJ/m^2$$

Thus, the total value of the J-integral is

$$J = J_e + J_p = 0.97 + 0.89 = 1.86\,MJ/m^2$$

Therefore, crack will grow because $J > J_{IC} = 1.20\,m.MPa$. Stable? Unstable? Let's find out how the crack will grow. Let

$$K_I = \alpha\sigma\sqrt{\pi a_e}\;; \qquad J_e = \frac{K_I^2}{E'} = \frac{(1-\nu^2)K_I^2}{E} = \frac{\pi a_e(1-\nu^2)(\alpha\sigma)^2}{E} \quad \text{and} \quad J_p = \alpha'\sigma_0\varepsilon_0\lambda_1 h_1\left(\frac{P}{P_0}\right)^{n+1}$$

so that the total value of the J-integral can be computed by

$$J = J_e + J_p = \frac{\pi a_e(1-\nu^2)(\alpha\sigma)^2}{E} + \alpha'\sigma_0\varepsilon_0\lambda_1 h_1\left(\frac{P}{P_0}\right)^{n+1}$$

If $a_e = 1.0617a$ and $\lambda_1 = (w-a)(a/w)$, then

$$J == \left[\frac{\pi(1-\nu^2)(\alpha\sigma)^2}{E}\right](1.0617a) + \left[\alpha'\sigma_0\varepsilon_0 h_1\left(\frac{P}{P_0}\right)^{n+1}\right](w-a)(a/w)$$

Thus, at constant load *dJ/da* is

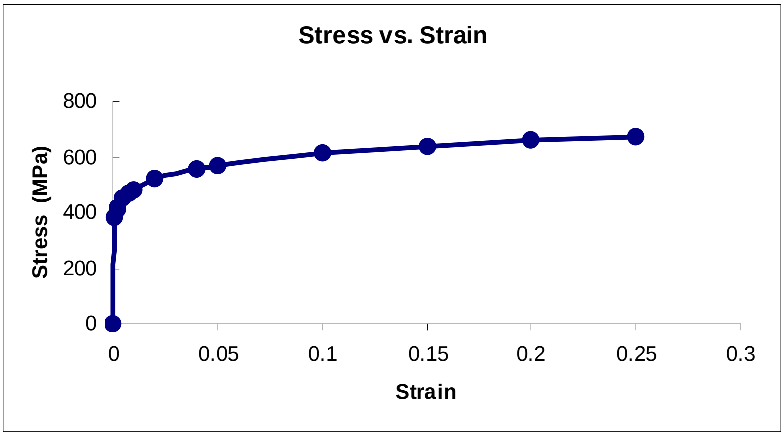
$$\frac{dJ}{da} = (1.0617)\left[\frac{\pi(1-\nu^2)(\alpha\sigma)^2}{E}\right] + \left[\alpha'\sigma_0\varepsilon_0 h_1\left(\frac{P}{P_0}\right)^{n+1}\right]\left[\frac{w-a}{w} - \frac{a}{w} + 1\right] = 26\,MPa$$

$$T_J = \frac{E}{\sigma_o^2}\frac{dJ}{da} = \frac{(207x10^3\,MPa)}{(414\,MPa)^2}(26\,MPa) = 31.40$$

Therefore, the crack will grow in a stable manner $T_J = 31.40 < T_R$

7.3 Repeat Problem 7.2 using the Hollomon equation $\sigma = K\varepsilon^n$ for the plastic region. Curve fitting should be performed using this equation for obtaining K and n. Assume plane strain conditions and make the necessary assumptions. Compare the result from Problem 7.2. What can you conclude from these results? Assume a contour as shown in Figure 7.6 with $y_{23} = 70\,mm$.

Solution: Single-edge cracked specimen



Data for plane strain condition:

$\sigma_{ys} = \sigma_o = 414\ MPa$ (From the above Figure) $a = 100\ mm$

$E = 207\ GPa$ $w = 400\ mm$

$v = 0.30$ $B = 150\ mm$

$J_{IC} = 1.20\ MPa \cdot m \cong 6.8\ \dfrac{in \cdot kips}{in^2}$ (From Ref. [8]) $L = 3w = 1{,}200\ mm$

$\sigma = 500\ MPa$ (Applied) $b = w - a = 300\ mm$ (Ligament)

Elastic Part:

For $a/w = 100/400 = 0.25$,

$$\alpha = 1.12 - 0.23(a/w) + 10.55(a/w)^2 - 21.71(a/w)^3 + 30.38(a/w)^4 = 1.5013$$

$$K_I = \alpha\sigma\sqrt{\pi a_e} = (1.5013)(500\ MPa)\sqrt{(\pi)(106.17x10^{-3}\ m)} = 469.10\ MPa\sqrt{m}$$

$$J_e = \frac{K_I^2}{E'} = \frac{(1 - v^2)K_I^2}{E} = \frac{(1 - 0.3^2)(469.10\ MPa\sqrt{m})^2}{207x10^3\ MPa} = 0.97\ m.MPa = 0.97\ MJ/m^2$$

Plastic part: Nonlinear regression using $\sigma = k\varepsilon^n$ yields $k = 776\ MPa$ and $n = 0.103$. The plastic strain energy density can be defined using the Hollomon equation as

$$\sigma = k\varepsilon^n \quad \text{and} \quad \varepsilon = \left(\frac{\sigma}{k}\right)^{1/n}$$

$$W_p = \int_0^{\varepsilon} \sigma \cdot d\varepsilon = \int_0^{\varepsilon} k\varepsilon^n d\varepsilon \qquad k = 776\ MPa \quad and \quad n = 0.103$$

$$W_p = \frac{k}{n+1}\varepsilon^{n+1} = \left(\frac{k}{n+1}\right)\left(\frac{\sigma}{K}\right)^{\frac{n+1}{n}} = \left(\frac{776\ MPa}{1.103}\right)\left(\frac{500\ MPa}{776\ MPa}\right)^{10.7087}$$

$$W_p = 6.35\ MPa = 6.35\ MJ/m^3$$

Assume a contour as shown in Figure 7.6 with $y_{23} = 70 mm$ so that $J_p = 2y_{23}W_p$ as per eq. (7.43), which represents a near-field condition. Thus,

$$J_p = 2y_{23}W_p = (2)(70x10^{-3}\ m)(6.35\ MPa) = 0.89\ m.MPa$$

$$J_I = J_e + J_p = 0.97 + 0.89 = 1.86\ MPa \cdot m = 1.86\ MJ/m^2$$

Crack growth occurs since $J > J_{IC}$. Note that if the value of y_{23} changes so will J_p and J_I. Anyway, both problems 7.2 and 7.3 (P7.2 and P7.3) give the same result as per chosen contour segment $y_{23} = 70\ mm$; otherwise $J_p(P7.2) \neq J_p(P7.3)$ or $J_I(P7.2) \neq J_I(P7.3)$.

Then, $K_I = \alpha\sigma\sqrt{\pi a}$ and $K_I^2 = \pi a(\alpha\sigma)^2$

$$J_e = \frac{K_I^2}{E'} = \frac{\pi(1-v^2)(\alpha\sigma)^2 a}{E}$$

$$W_p = \frac{K}{n+1}\varepsilon^{n+1} = \left(\frac{K}{n+1}\right)\left(\frac{\sigma}{K}\right)^{\frac{n+1}{n}}$$

$$J_p = 2y_{23}W_p = \frac{K}{n+1}\varepsilon^{n+1} = 2y_{23}\left(\frac{K}{n+1}\right)\left(\frac{\sigma}{K}\right)^{\frac{n+1}{n}}$$

$$J_I = J_e + J_p = \frac{\pi(1-v^2)(\alpha\sigma)^2 a}{E} + 2y_{23}\left(\frac{K}{n+1}\right)\left(\frac{\sigma}{K}\right)^{\frac{n+1}{n}}$$

$$\frac{dJ_I}{da} = \frac{\pi(1-v^2)\alpha^2\sigma^2}{E} = \frac{\pi(1-0.3^2)(1.5013x500)^2}{207x10^3}$$

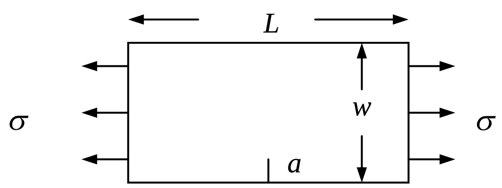
$$\frac{dJ_I}{da} = 7.78\ MPa = 7.78\ MJ/m^3$$

$$T_J = \frac{E}{\sigma_o^2}\frac{dJ}{da} = \frac{(207x10^3\ MPa)}{(414\ MPa)^2}(7.78\ MPa) = 9.40$$

Therefore, the crack will grow in a stable manner $T_J = 9.40 < T_R$.

7.4 A steel plate having a single-edge crack is loaded as shown below. Calculate **(a)** the load-line displacement (μ) and **(b)** the crack opening displacement (δ) that corresponds to a point on the resistance curve (not included). Assume plane strain conditions and use the following data: $v = 0.3$, $h_1 = 0.523$, $h_2 = 1.93$, $h_3 = 3.42$, $B = 0.15\ m$, $w = 0.40\ m$, $L = 1.2\ m$ and $a = 0.1\ m$. $J_{IC} = 1.20\ kJ/m^2$, $\sigma = 500\ MPa$ and $E = 206{,}850\ MPa$

Solution:



(a) From Table 7.2,

$$J_p = \alpha'\sigma_0\varepsilon_0(w-a)h_1\left(\frac{P}{P_0}\right)^{n+1}$$

$$\mu_p = \alpha' a\varepsilon_0 h_3\left(\frac{P}{P_0}\right)^{n} \quad \text{so that} \quad \left(\frac{P}{P_0}\right) = \left[\frac{\mu_p}{\alpha' a\varepsilon_0 h_{31}}\right]^{\frac{1}{n}}$$

Eliminating P/P_o yields

$$J_p = \alpha'\sigma_0\varepsilon_0(w-a)h_1\left(\frac{\mu_p}{\alpha' a\varepsilon_0 h_3}\right)^{(n+1)/n}$$

$$J_p = \frac{\sigma_0 h_1(w-a)}{(\alpha'\varepsilon_0)^{1/n}}\left(\frac{\mu_p}{a h_3}\right)^{(n+1)/n}$$

$$\mu_p = a h_3(\alpha'\varepsilon_0)^{1/(n+1)}\left(\frac{J_p}{\sigma_0\lambda_1 h_1}\right)^{\frac{n}{n+1}}$$

Similarly,

$$\delta = \alpha' a\varepsilon_0 h_2\left(\frac{P}{P_0}\right)^{n} = a h_2(\alpha'\varepsilon_0)^{1/(n+1)}\left(\frac{J_p}{\sigma_0\lambda_1 h_1}\right)^{\frac{n}{n+1}}$$

These are the equations to be used in this problem. However, we need the value of J_p at $\sigma = 500\ MPa$. Let's get.

Data for plane strain condition:

$\sigma_{ys} = \sigma_o = 414\ MPa$ (From the above Figure) $a = 100\ mm \quad B = 150\ mm$

$E = 207\ GPa \qquad v = 0.30$ $w = 400\ mm$

$J_{IC} = 1.20\ MPa \cdot m \cong 6.8\ \dfrac{in \cdot kips}{in^2}$ (From Ref. [8]) $L = 3w = 1{,}200\ mm$

$\sigma = 500\ MPa$ (Applied) $b = w - a = 300\ mm$ (Ligament)

Elastic Part:

Nonlinear regression using $\varepsilon = \alpha' \varepsilon_0 \left(\dfrac{\sigma}{\sigma_0} \right)^n$ yields $\alpha' = 1.12$ and $n = 9.71$. From Table 7.2,

$$P = w\sigma = \left(400x10^{-3}\ m\right)\left(500\ MPa\right) = 200\ MN/m$$

$$x = \left[1 + \left(\frac{a}{w-a}\right)^2\right]^{1/2} - \frac{a}{w-a} = \left[1 + \left(\frac{100}{400-100}\right)^2\right]^{1/2} - \frac{100}{400-a}100 = 0.82137$$

$$P_0 = 1.455x(w-a)\sigma_0 = (1.455)(0.82137)\left(300x10^{-3}\ m\right)\left(414\ MPa\right) = 148.43\ MN/m$$

$$\lambda_1 = (w-a)(a/w) = (400-100)(100/400) = 75\ mm$$

Also, $\beta = 6$ and

$$r = r_y = \left(\frac{1}{\pi\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{K_I}{\sigma_0}\right)^2 = \left(\frac{1}{\pi\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma\sqrt{\pi a_e}}{\sigma_0}\right)^2 = \left(\frac{a_e}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_0}\right)^2$$

$$\overline{w} = \frac{1}{1 + (P/P_0)^2}$$

The crack size can be calculated using

$$a_e = a + \overline{w} r_y$$

$$a_e = a + \left[\frac{1}{1 + (P/P_0)^2}\right]\left(\frac{1}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_0}\right)^2 a_e = a + \left[\frac{1}{1 + (200/148.43)^2}\right]\left(\frac{1}{6}\right)\left(\frac{9.71-1}{9.71+1}\right)\left(\frac{500}{414}\right)(a_e)$$

$$a_e = a + 0.05814 a_e$$

$$a_e = 1.0617a$$

$$a = 100\ mm$$

$$a_e = 1.0617(100\ mm) = 106.17\ mm$$

For $a/w = 100/400 = 0.25$,

$$\alpha = 1.12 - 0.23(a/w) + 10.55(a/w)^2 - 21.71(a/w)^3 + 30.38(a/w)^4 = 1.5013$$

$$K_I = \alpha\sigma\sqrt{\pi a_e} = (1.5013)(500\ MPa)\sqrt{(\pi)(106.17x10^{-3}\ m)} = 469.10\ MPa\sqrt{m}$$

$$J_e = \frac{K_I^2}{E'} = \frac{(1-\nu^2)K_I^2}{E} = \frac{(1-0.3^2)(469.10\ MPa\sqrt{m})^2}{207x10^3\ MPa} = 0.97\ m.MPa = 0.97\ MJ/m^2$$

$$J_e = 0.97\ MJ/m^2$$

From Problem 7.2,

$$r = r_y = \left(\frac{a_e}{\beta}\right)\left(\frac{n-1}{n+1}\right)\left(\frac{\sigma}{\sigma_0}\right)^2 = \left(\frac{106.17\ mm}{6}\right)\left(\frac{9.71-1}{9.71+1}\right)\left(\frac{500}{414}\right)^2 = 21\ mm$$

From eq. (7.51) and Table 7.2,

$$P = w\sigma = (400x10^{-3}\ m)(500\ MPa) = 200\ MN/m$$

$$x = \left[1 + \left(\frac{a}{w-a}\right)^2\right]^{1/2} - \frac{a}{w-a} = \left[1 + \left(\frac{100}{400-100}\right)^2\right]^{1/2} - \frac{100}{400-a}100 = 0.82137$$

$$P_0 = 1.455x(w-a)\sigma_0 = (1.455)(0.82137)(300x10^{-3}\ m)(414\ MPa) = 148.43\ MN/m$$

$$\lambda_1 = (w-a)(a/w) = (400-100)(100/400) = 75\ mm$$

$$h_1 = 0.523, \quad \alpha' = 1.12 \quad \text{and} \quad \lambda_1 = 75\ mm = 75x10^{-3}\ m$$

$$J_p = \alpha'\sigma_0\varepsilon_0\lambda_1 h_1\left(\frac{P}{P_0}\right)^{n+1} \quad with \quad \varepsilon_0 = \sigma_0/E = 0.002$$

$$J_p = (1.12)(414\ MPa)(0.002)(75x10^{-3}\ m)(0.523)\left(\frac{200}{148.43}\right)^{1+9.71}$$

$$J_p = 0.89\ MJ/m^2$$

$$J = J_e + J_p = 0.97\ MJ/m^2 + 0.89\ MJ/m^2 = 1.86\ MJ/m^2$$

Thus,

$$\mu_p = ah_3(\alpha'\varepsilon_0)^{1/(n+1)}\left(\frac{J_p}{\sigma_0\lambda_1 h_1}\right)^{\frac{n}{n+1}}$$

$$\mu_p = (0.1\,m)(3.42)[(1.12)(0.002)]^{0.0934}\left[\frac{0.89\,m.MPa}{(414\,MPa)(0.3\,m)(0.523)}\right]^{0.9066}$$

$$\mu_p = 3.90\,mm$$

(b) The crack opening displacement is

$$\delta = \alpha' a\varepsilon_0 h_2\left(\frac{P}{P_0}\right)^n = ah_2(\alpha'\varepsilon_0)^{1/(n+1)}\left(\frac{J_p}{\sigma_0\lambda_1 h_1}\right)^{\frac{n}{n+1}}$$

$$\delta = (0.1\,m)(1.93)[(1.12)(0.002)]^{0.0934}\left[\frac{0.89\,m.MPa}{(414\,MPa)(0.3\,m)(0.523)}\right]^{0.9066}$$

$$\delta = 2.20\,mm$$

Therefore, $\mu_p = 3.90\,mm$ and $\delta = 2.20\,mm$ at the onset of crack growth since $J_{IC} = 1.2\ MN/m^2$ and crack blunting required these values.

7.5 Determine the strain hardening exponent (n) for a steel with σ_{ys} = 400 MPa, E = 207 GPa. Assume that it obeys the Hollomon equation $\sigma_p = k\varepsilon^n$. Consider the maximum plastic stress in your calculations.

Solution:

$\sigma_p = \sigma_{\max}\ @\ \varepsilon = \varepsilon_p = 1 = \varepsilon_{\max}$ and $\sigma_{max} = k = 700\ MPa$

@ yielding, $\sigma_{ys} = k\varepsilon_{ys}^n$, $\sigma_{\max} = k\varepsilon_{\max}^n$. and $\varepsilon_{ys} = \dfrac{\sigma_{ys}}{E} = 1.93\times10^{-3}$

$$\frac{\sigma_{ys}}{\sigma_{\max}} = \frac{k\varepsilon_{ys}^n}{k\varepsilon_{\max}} = \varepsilon_{ys}^n$$

$$n = \frac{\ln\left(\dfrac{\sigma_{@\,\varepsilon_{ys}}}{\sigma_{\max}}\right)}{\ln(\varepsilon_{ys})} = \frac{\ln\left(\dfrac{400}{700}\right)}{\ln(1.93\times10^{-3})} \cong 0.0895$$

7.6 (a) Derive an expression for the J-integral $J_p/J_e = f(\sigma(\sigma_o)$ ratio as per Rice model. **(b)** Plot the resultant expression for a remote stress to yield stress ratio range $0 \le \sigma/\sigma_o \le 1$. Here, $\sigma_o = \sigma_{ys}$. Explain the resultant trend.

Solution:

(a) The plastic case:

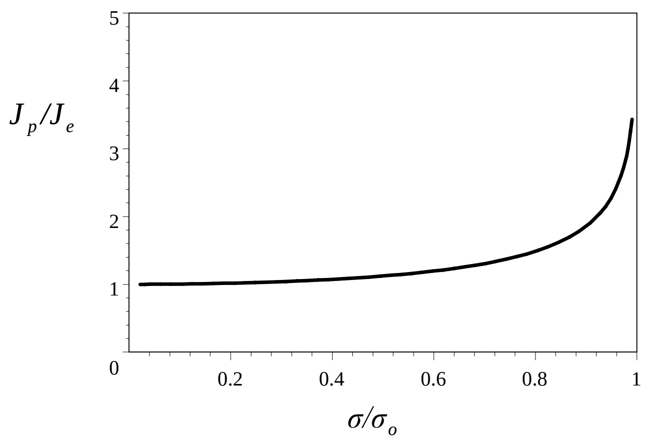
$$\delta_t = \frac{(k+1)a\sigma_o}{\pi G} log\left[sec\left(\frac{\pi\sigma}{2\sigma_o}\right)\right]$$

$$J_p = \sigma_o \delta_t = \frac{(k+1)a\sigma_o^2}{\pi G} log\left[sec\left(\frac{\pi\sigma}{2\sigma_o}\right)\right]$$

The elastic case:

$$\left(\begin{array}{l} J_e = \frac{(k+1)K_I^2}{8G} = \frac{(k+1)\pi a\sigma^2}{8G} \\ \frac{J_p}{J_e} = \frac{2\log\left[\sec\left(\frac{\pi\sigma}{2\sigma_o}\right)\right]}{\left(\frac{\pi\sigma}{2\sigma_o}\right)^2} = f(\sigma/\sigma_o) \end{array}\right)$$

From the plot, $J_p/J_e \to \infty$ as $\sigma/\sigma_o \to 1$ since $\sec(\pi/2) \to \infty$ and $\sigma \to \sigma_o$.



They significantly differ from each other due to a large plasticity. At $\sigma/\sigma_o > 0.8$, J_P dominates.

7.7 Calculate the total J-integral (J) for a 2024 Al-alloy plate containing a single-edge crack under plane stress conditions. Plot b) the $\sigma = f(\varepsilon)$ and $J_I = f(\sigma)$. Use the following data to carry out all calculations: a = 1.40 mm, w = 19 mm, B = 0.8 mm, L = 10 cm, $\sigma_o = 64\ MPa$, $E = 72{,}300\ MPa$, $\alpha' = 0.35$, n = 5 and F = 1.01 kN.

Solution:

Elastic part:
a) The following calculations are self-contained. Thus,

$$\epsilon_o = \frac{\sigma_o}{E} = \frac{64}{72{,}300} = 8.852 \times 10^{-4} \qquad A = wB = 15.20x10^{-6}\ m^2$$

$$\sigma = \frac{F}{A} = \frac{1.01x10^{-3}\ MN}{15.20x10^{-6}\ m^2} = 66.45\ MPa > \sigma_{ys}$$

For a single-edge crack,

$x = a/w = 1.4/19 = 0.073684$

$\alpha = 1.12 - 0.23(0.073684) + 10.55(0.073684)^2 - 21.71(0.073684)^3 = 1.1516$

$$\alpha = 1.12 - 0.23x + 10.55x^2 - 21.71x^3 = 1.1516$$

$$J_e = \frac{K_I^2}{E} = \frac{\pi a \alpha^2 \sigma^2}{E} = \frac{\pi(1.40x10^{-3}\ m)(1.1516)^2(66.45\ MPa)^2}{72{,}300\ MPa}$$

$$J_e = 3.5623 \times 10^{-4}\ MPa.m = 356.23\ J/m^2$$

$$J_e = 356.23\ J/m^2$$

Plastic part:
From Table 7.2 and eqs. (7.60),

$$\lambda_1 = (w-a)(a/w) = (19-1.4)(1.4/19) = 1.2968\ mm$$

$$\eta = \sqrt{1 + \left(\frac{a}{w-a}\right)^2} - \frac{a}{w-a} = \sqrt{1 + \left(\frac{1.4}{19-1.4}\right)^2} - \frac{1.4}{19-1.4} = 0.92361$$

$$P_o = 1.072\eta(w-a)\sigma_o = 1.072 * 0.92361 * (19-1.4) * 10^{-3} * (64) = 1.1153\ MN/m$$

$$P = w\sigma = (19 * 10^{-3}\ m)(66.45\ MPa) = 1.2626\ MN/m$$

$$h_1 = \frac{3\pi a\sqrt{n}}{4\psi^{1-n}\lambda_1}\left(\frac{\sigma}{\sigma_o}\right)^{n+1}\left(\frac{P_o}{P}\right)^{n+1}$$

$$h_1 = \frac{3\pi(1.4)\sqrt{5}}{4(1)^{1-n}(1.2968)}\left(\frac{66.45}{64}\right)^{5+1}\left(\frac{1.1153}{1.2626}\right)^{5+1} = 3.3853$$

From eq. (7.54a),

$$J_p = \alpha'\sigma_o\epsilon_o\lambda_1 h_1\left(\frac{P}{P_o}\right)^{n+1} = 0.35 * 64 * 8.852 \times 10^{-4} * (1.2968 * 10^{-3}) * 3.3853\left(\frac{1.2626}{1.1153}\right)^{5+1}(10^6)$$

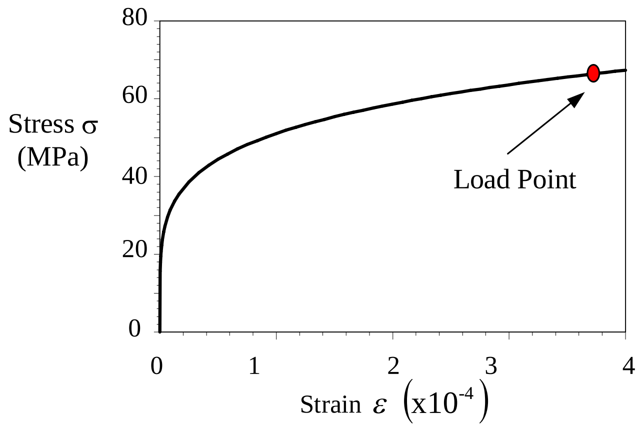
$$J_p = 183.23\ J/m^2$$

$$J_I = J_e + J_p = 356.23\ J/m^2 + 183.23\ J/m^2 = 539.46\ J/m^2$$

(b) These results indicate that $J_p = 0.34J$ and $J_e = 0.66J$. Therefore, $J_e > J_p$ and contributes 66% to J.

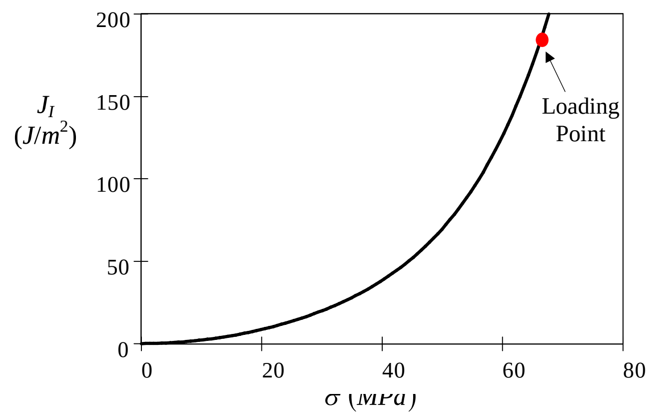
(c) The required stress-strain curve can be determined using eq. (7.30). Thus,

$$\sigma = \sigma_{ys}\left(\frac{\epsilon}{\alpha'\epsilon_{ys}}\right)^{1/n} = (322.08\ MPa)\sqrt[5]{\epsilon}$$



The J-integral plot along with the load point is based on the following equation:

$$J_I = 2.1247 \times 10^{-2}\sigma^2 + 1.0638 \times 10^{-9}\sigma^6$$

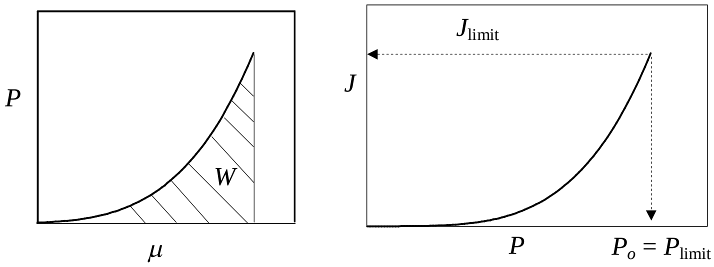


The trend of the $J_I = f(\sigma)$ resembles the trend shown in Figure 7.5.

7.8 The plastic J-integral (J_p) for some configurations can be defined by $J = \eta W/(Bb),$ where $W =$ absorbed energy, Bb = cross sectional area, $b = (w - a)$ = ligament and $\eta =$ constant. This integral can then be separated into elastic and plastic components. For pure tension,

$$J = J_e + J_p = \frac{K_I^2}{E'} + \frac{\eta_p W_p}{Bb} \tag{1}$$

Consider a strain hardenable material and a specimen with unit thickness B. If the plastic load is defined by $P = C\mu_p^n$, where C is the compliance and n is the strain hardening exponent, then the load-displacement and J-P profiles are schematically shown below.



Derive an expression for η_p.

Solution:

The area under the curve is the strain energy density define by

$$W_p = \int_o^{\mu_p} P d\mu_p = \int_o^{\mu_p} C\mu_p^n d\mu_p = \frac{n}{n+1} C\mu_p^{n+1} = \frac{n}{n+1} P\mu_p \quad \text{since } P = C\mu_p^n \tag{2}$$

Since W_p is a maximum energy then P reaches P_o (the load limit). Thus,

$$W_p = \frac{n}{n+1} P_o \mu_p \tag{3}$$

From eq. (7.56), $\mu_p = \alpha' a \varepsilon_o h_3 \left(\frac{P}{P_o}\right)^n$ and

$$W_p = \frac{n}{n+1} P_o \alpha' a \varepsilon_o h_3 \left(\frac{P}{P_o}\right)^n \tag{4}$$

Substituting this expression into the J-integral equation above yields the plastic part as

$$J_p = \frac{\eta_p}{Bb} \frac{n}{n+1} P_o \alpha' a \varepsilon_o h_3 \left(\frac{P}{P_o}\right)^n \tag{5}$$

Equate eq. (7.54) and (5) and solve for η_p

$$\frac{\eta_p}{b}\frac{n}{n+1}P_o\alpha' a\varepsilon_o h_3\left(\frac{P}{P_o}\right)^n = \alpha'\varepsilon_o\sigma_o\lambda_1 h_1\left(\frac{P}{P_o}\right)^n$$

$$\eta_p = \frac{n+1}{n}\frac{b^2}{a}\frac{\sigma_o}{P_o}\frac{h_1}{h_3}$$

CHAPTER 8

MIXED-MODE FRACTURE MECHANICS

8.1 A large plate (2024-0 Al-alloy) containing a central crack is subjected to a combined mode I-II loading. The internal stresses are σ_y = 138 MPa and τ_{xy} = 103 MPa. Use the Maximum Principal Stress Criterion (σ_θ-criterion) to calculate the fracture toughness (K_{IC}). Use the following data: crack length $2a = 76$ mm, $\nu = 1/3$, and E = 72,300 MPa.

Solution:

a = 38 mm, σ = 138 MPa, τ = 103 MPa

$$K_I = \sigma_y \sqrt{\pi a} = 47.68\ MPa\sqrt{m}$$
$$K_{II} = \tau_{xy} \sqrt{\pi a} = 35.60\ MPa\sqrt{m}$$
$$K_I / K_{II} = 1.3393$$

Using eq. (8.29a) yields

$$\theta_0 = -ar\cos\left[\frac{1}{3}\left(1 - \frac{x\left(x - 3\sqrt{x^2+8}\right)}{x^2+9}\right)\right] = -48.60^o$$

From eq. (8.33),

$$K_{IC} = K_I \cos^3\frac{\theta_0}{2} - 3K_{II}\cos^2\frac{\theta_0}{2}\sin\frac{\theta_0}{2}$$
$$K_{IC} = 72.50\ MPa\sqrt{m}$$

From eq. (8.34),

$$K_{IIC} = \sqrt{\frac{3}{4}} \cdot K_{IC} = 62.77\ MPa\sqrt{m}$$

8.2 Repeat problem 8.1 using the Strain Energy Density Factor (S) Criterion (S-criterion).

Solution:

$$K_I = \sigma\sqrt{\pi a} = 47.68\ MPa\sqrt{m}$$
$$K_I = \tau_{xy}\sqrt{\pi a} = 35.60\ MPa\sqrt{m}$$
$$K_{II}/K_I = 0.75$$

Use eq. (8.53) to solve for the fracture angle

$$\sin\theta_0[2\cos\theta_0 - k + 1]K_I^2 + \left[4\cos^2\theta_0 - (k-1)\cos\theta_0 - 2\right]K_I K_{II} - \sin\theta_0[6\cos\theta_0 - k + 1]K_{II}^2 = 0$$
$$\theta = -48.10^o$$

Using Equation (8.54) yields

$$K_{IC}^2 = \frac{4E}{(k-1)(1+\nu)}\left[a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2\right]$$
$$K_{IC} = 59.35 MPa\sqrt{m}$$

From eq. (8.57) and (8.51),

$$K_{IIC} = 0.9045 K_{IC} = 53.68\ MPa\sqrt{m}$$

$$S_c = 14.26\ kPa\cdot m = 14.26\ kJ/m^2$$

8.3 Determine **(a)** the incline angle β and **(b)** the applied tension σ in Example 8.1.

Solution:

(a) $K_I = \sigma\sqrt{\pi a}\cdot\sin^2\beta$

$$K_{II} = \sigma\sqrt{\pi a}\cdot\sin\beta\cos\beta$$

$$\frac{K_I}{K_{II}} = tan\,\beta = \frac{47.68 MPa\sqrt{m}}{35.60 MPa\sqrt{m}}$$
$$\beta = 53.25^o$$

(b) $K_I = \sigma\sqrt{\pi a}\sin^2\beta$ a = 38 mm

$$\sigma = \frac{K_I}{\sqrt{\pi a}\cdot\sin^2\beta} = \frac{47.68 MPa\sqrt{m}}{\sqrt{\pi\left(38\times10^{-3}m\right)\sin^2\left(53.25^o\right)}}$$

$$\sigma = 214.95\ MPa$$

8.4 Calculate the critical stress (fracture stress) for the problem described in Problem 8.1 according to the σ_θ - criterion. Will crack propagation take place?

Solution:

From problem 8.1 & 8.2:

$$\beta = 53.25^o$$
$$\sigma \approx 215 MPa \quad \text{(Applied)}$$

From Example 8.4:

a = 38 mm θ_o = -48.60º

E = 72,300 MPa ν = 1/3

From eq. (8.35),

$$\sigma_c = \frac{K_{IC}}{\sin\beta\sqrt{\pi a}}\left[b_{11} - 2b_{12}\cos\beta + b_{22}\cos^2\beta\right]_{\theta=\theta_0}^{-1/2} = 492\ MPa$$

where

$$b_{11} = \frac{1}{8}\left(1+\cos\theta_0\right)^3 = 0.57314$$

$$b_{12} = \frac{3}{8}\left(\sin\theta_0\right)\left(1+\cos\theta_0\right)^2 = -0.77635$$

$$b_{22} = \frac{9}{8}\sin^2\theta_0\left(1+\cos\theta_0\right) = 1.05161$$

Crack propagation will not take place since $\sigma < \sigma_c$.

8.5 Repeat problem 8.4 using the S-criterion with $\sigma = 215\ MPa$.

Solution:

$$K_I = \sigma\sqrt{\pi a}\cdot\sin^2\beta \qquad K_I = \sigma^2\pi a\cdot\sin^4\beta \qquad K_I K_{II} = \sigma^2\pi a\cdot\sin^2\beta\cos\beta$$

$$K_{II} = \sigma\sqrt{\pi a}\cdot\sin\beta\cos\beta \qquad K_{II} = \sigma^2\pi a\cdot\sin^2\beta\cos^2\beta$$

From eq. (8.54),

$$\sigma_c = \frac{K_{IC}}{sin\,\beta\sqrt{\pi a}}\sqrt{\frac{(\kappa-1)(1+\nu)}{4E}}\cdot\left[a_{11}\,sin^2\beta + 2a_{12}\,cos\,\beta + a_{22}\,cos^2\beta\right]_{\theta=\theta_o}^{-1/2}$$

where

$$a_{11} = \frac{1}{16G}\left[\left(1+cos\theta\right)\left(k-cos\theta\right)\right] = \frac{(1+\nu)}{8E}\left[\left(1+cos\theta\right)\left(k-cos\theta\right)\right] = 3.84x10^{-6}$$

$$a_{12} = \frac{1}{16G}sin\theta\cdot\left[2cos\theta-(k-1)\right] = \frac{(1+\nu)}{8E}sin\theta\cdot\left[2cos\theta-(k-1)\right] = -1.15x10^{-6}$$

$$a_{22} = \frac{(1+\nu)}{8E}\left[(k+1)\left(1-cos\theta\right)+\left(1+cos\theta\right)\left(3cos\theta-1\right)\right] = 5.90x10^{-6}$$

Thus,

$\sigma_c \approx 204\ MPa$

$\sigma \approx 215\ MPa$ (Applied)

Therefore, crack propagation will occur because $\sigma > \sigma_c$.

8.6 Show that the stress intensity factor is $K_I = \sqrt{8/11}K_{IC}$ when $K_I = 2K_{III}$ and that the Poisson's ration is $v = 1/3$.

Solution:

From eq. (8.13) with $E' = E/(1-v^2)$ and $K_{II} = 0$,

$$K_{IC}^2 = K_I^2 + \frac{E'(1+v)}{E}K_{III}^2 \quad with \quad K_I = 2K_{III}$$

$$K_{IC}^2 = K_I^2 + \frac{E(1+v)}{E(1-v^2)}K_{III}^2 = K_I^2 + \frac{(1+v)}{(1+v)(1-v)}K_{III}^2$$

$$K_{IC}^2 = K_I^2 + \frac{K_{III}^2}{(1-v)} = K_I^2 + \frac{K_I^2}{4(1-v)} = \left[1 + \frac{1}{4(1-v)}\right]K_I^2$$

$$K_{IC}^2 = \left[1 + \frac{1}{4(1-1/3)}\right]K_I^2 = \left[\frac{11}{8}\right]K_I^2$$

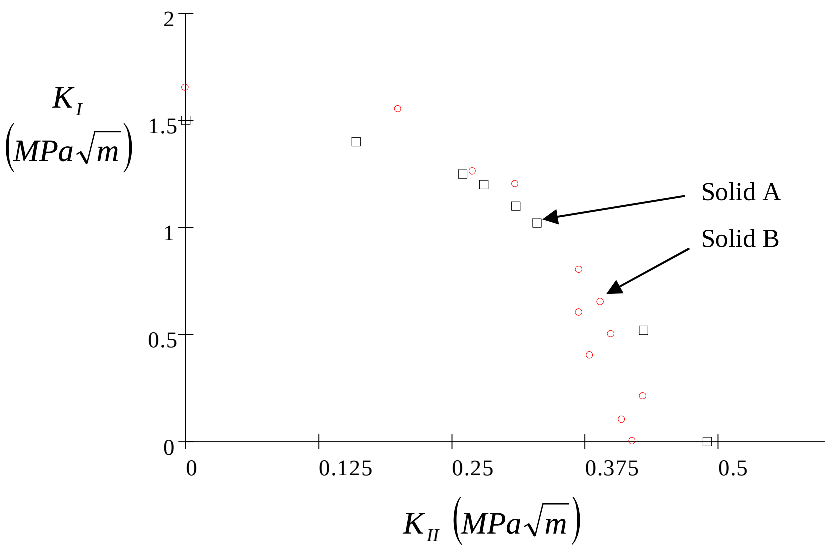
Thus,

$$K_I = \sqrt{\frac{8}{11}} \cdot K_{IC}$$

8.7 Use the experimental mixed mode fracture data or Solid A and Solid B given below to compare the mixed mode fracture criteria discussed in this chapter. Determine which criterion is the most suited to predict the mixed mode fracture behavior of these solids.

Solid A		Solid B	
K_{II}	K_I	K_{II}	K_I
(MPa$\sqrt{m}$)	(MPa$\sqrt{m}$)	(MPa$\sqrt{m}$)	(MPa$\sqrt{m}$)
0.49	0	0.42	0
0.43	0.52	0.41	0.10
0.33	1.02	0.43	0.21
0.31	1.10	0.38	0.40
0.28	1.20	0.40	0.50
0.26	1.25	0.37	0.60
0.16	1.40	0.39	0.65
0	1.50	0.37	0.80
		0.31	1.20
		0.27	1.26
		0.20	1.55
		0	1.65

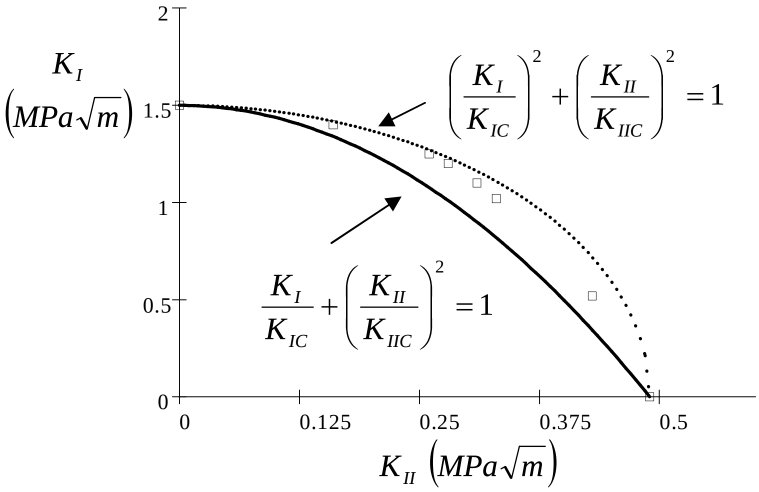
Solution:
Plot the given data and use all possible mixed mode criteria in order to determine the most suited criterion.



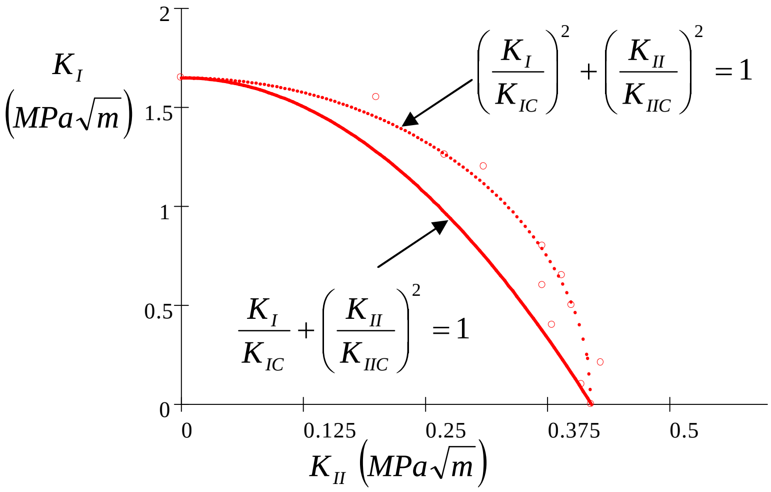
From the given data, Solid A: $K_{IC} = 1.50\ MPa\sqrt{m}$ & $K_{IIC} = 0.49\ MPa\sqrt{m}$

Solid B: $K_{IC} = 1.65\ MPa\sqrt{m}$ & $K_{IIC} = 0.42\ MPa\sqrt{m}$

For Solid A,

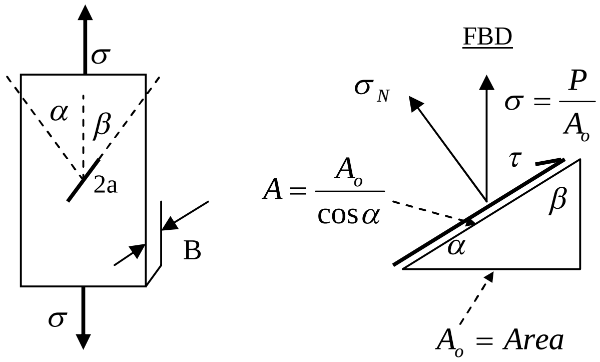


For Solid B,



Therefore, the criterion $\left(\frac{K_I}{K_{IC}}\right)^2 + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1$ fits both data sets better than $\frac{K_I}{K_{IC}} + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1$.

8.8 A large and wide plate has a small through-the-thickness central crack as shown in the figure below. Use the free-body diagram (FBD) to derive expressions for **(a)** the normal (σ_N) and shear (τ) stresses and **(b)** the intensity factors K_I and K_{II} as functions of β.



Solution:

(a) Using the FBD yields the normal and shear stresses

$$\sum F_y = \sigma_N A - \sigma A_o \cos\alpha = 0$$

$$\sigma_N \frac{A_o}{\cos\alpha} - \sigma A_o \cos\alpha = 0$$

$$\sigma_N = \sigma \cos^2 \alpha = \sigma \sin^2 \beta$$

$$\sum F_x = \tau A - \sigma A_o \cos\beta = 0$$

$$\tau \frac{A_o}{\cos\alpha} - \sigma A_o \cos\beta = 0$$

$$\tau = \sigma \cos(\alpha)\cos(\beta) = \sigma \sin(\beta).\cos(\beta)$$

(b) The stress intensity factor equations are

$$K_I = \sigma_N \sqrt{\pi a} =$$

$$K_I = \sigma\sqrt{\pi a}\,\sin^2(\beta)$$

$$K_{II} = \tau\sqrt{\pi a}$$

$$K_{II} = \sigma\sqrt{\pi a}\,\sin(\beta)\cos(\beta)$$

8.9 Two identical cracked plates as shown in problem 8.8 are to be tested to determine K_{IC} and K_{IIC}. The observed critical tension loads and the incline angles were 120 MPa @ $\alpha = 0$ and 130 MPa @ $\alpha = \pi/4$, respectively. Use the equations given below to determine the fracture toughness for mode I and II.

$$\frac{K_I}{K_{IC}} + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1 \quad \text{and} \quad \left(\frac{K_I}{K_{IC}}\right)^2 + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1$$

Solution:

For case $\sigma = 120\ MPa$ @ $\alpha = 0$ and $\beta = \pi/2 = 90^o$,

$$K_I = \sigma\sqrt{\pi a}\sin^2(\beta) = (120\ MPa)\sqrt{\pi(20x10^{-3}\ m)}(1) = 30\ MPa\sqrt{m}$$

$$K_{II} = \sigma\sqrt{\pi a}\sin(\beta)\cos(\beta) = 0\ MPa\sqrt{m}$$

$$\frac{K_I}{K_{IC}} + \left(\cancelto{0}{\frac{K_{II}}{K_{IIC}}}\right)^2 = 1 \qquad K_{IC} = K_I = 30\ MPa\sqrt{m}$$

For case $\sigma = 130\ MPa$ @ $\alpha = \pi/4$ and $\beta = \pi/4 = 45^o$,

$$K_I = \sigma\sqrt{\pi a}\sin^2(\beta) = (130\ MPa)\sqrt{\pi(20x10^{-3}\ m)}\left(\frac{\sqrt{2}}{2}\right)^2 = 16.30\ MPa\sqrt{m}$$

$$K_{II} = \sigma\sqrt{\pi a}\sin(\beta)\cos(\beta) = (130\ MPa)\sqrt{\pi(20x10^{-3}\ m)}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = 16.30\ MPa\sqrt{m}$$

$$\frac{K_I}{K_{IC}} + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1 \qquad K_{IC} = 30\ MPa\sqrt{m}$$

$$K_{IIC} = K_{II}\left(1 - \frac{K_I}{K_{IC}}\right)^{-1/2}$$

$$K_{IIC} = (16.30\ MPa\sqrt{m})(1 - 16.30/30)^{-1/2}$$

$$K_{IIC} = 24.12\ MPa\sqrt{m}$$

$$\left(\frac{K_I}{K_{IC}}\right)^2 + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1; \quad K_{IC} = 30\ MPa\sqrt{m}$$

$$K_{IIC} = K_{II}\left[1 - \left(\frac{K_I}{K_{IC}}\right)^2\right]^{-1/2}$$

$$K_{IIC} = (16.30\ MPa\sqrt{m})\left[1 - (16.30/30)^2\right]^{-1/2}$$

$$K_{IIC} = 19.42\ MPa\sqrt{m}$$

These results indicate that the care must be taken in selecting a particular fracture criterion. Nevertheless, the fracture criterion $\left(\frac{K_I}{K_{IC}}\right)^2 + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1$ gives conservative values for K_{IIC} when compared to $\frac{K_I}{K_{IC}} + \left(\frac{K_{II}}{K_{IIC}}\right)^2 = 1$

CHAPTER 9

FATIGUE CRACK GROWTH

9.1 Show that Paris equation can take the form $\frac{da}{dN} = A\left(\frac{E\sigma_{ys}}{1-v^2}\right)^{n/2} (\Delta\delta_t)^{n/2}$, where $\Delta\delta_t$ is the change of crack tip opening displacement, E is the modulus of elasticity, and A is a constant.

Solution:

From eq. (7.45) or (8.10) with $K_{II} = K_{III} = 0$

$$J = \frac{K_I^2}{E'} = \frac{(1-v^2)K_I^2}{E} \qquad K_I = \sqrt{\frac{EJ}{1-v^2}} \qquad \Delta K = \sqrt{\frac{E\Delta J}{1-v^2}}$$

Substituting ΔK into Paris expression, eq. (9.6) yields

$$\frac{da}{dN} = A(\Delta K)^n = A\left[\sqrt{\frac{E\Delta J}{1-v^2}}\right]^n$$

From eq. (6.66), $J = \sigma_{ys}\delta_t$ and $\Delta J = \sigma_{ys}\Delta\delta_t$. Thus,

$$\frac{da}{dN} = A\left(\frac{E\sigma_{ys}}{1-v^2}\right)^{n/2} (\Delta\delta_x)^{n/2}$$

9.2 **(a)** Show that $\frac{da}{dN} = C(\Delta K_e)^n$, where $C = \frac{A}{(1-R)^{n(1-\alpha)}}$ and $\Delta K_e = K_{\max}(1-R)^\alpha$ is Walker's effective stress intensity factor range. **(b)** Plot $\frac{da}{dN} = C(\Delta K)^n$ and $\frac{da}{dN} = A(\Delta K)^n$ for a 4340 steel having $\sigma_{ys} = 1{,}254\ MPa$, $\sigma_{ts} = 1{,}296\ MPa$, $K_{IC} = 130\ MPa\sqrt{m}$, $\alpha = 0.42$, $n = 3.24$, $R = 0.7$, and $A = 5.11x10^{-11}$..

Solution

(a) $\Delta K_e = K_{\max}(1-R)^\alpha$

$$R = \frac{K_{\min}}{K_{\max}} = \frac{K_{\max} - K_{\max} + K_{\min}}{K_{\max}} = \frac{K_{\max} - \Delta K}{K_{\max}} = 1 - \frac{\Delta K}{K_{\max}}$$

$$\Delta K = K_{\max}(1-R)$$
$$K_{\max} = \frac{\Delta K}{1-R}$$

Then,

$$\Delta K_e = \frac{\Delta K}{(1-R)^{1-\alpha}} \quad \text{and}$$

$$\frac{da}{dN} = A(\Delta K_e)^n = A\left[\frac{\Delta K}{(1-R)^{1-\alpha}}\right]^n = \frac{A}{(1-R)^{n(1-\alpha)}}\left[\Delta K\right]^n$$

where

$$C = \frac{A}{(1-R)^{n(1-\alpha)}} \qquad \text{For } R \neq 1$$

Thus,

$$\frac{da}{dN} = C(\Delta K)^n$$

If $R = 0$, then $C = A$.

(b) If $\sigma_{ys} = 1{,}254\ MPa$, $\sigma_{ts} = 1{,}296\ MPa$, $K_{IC} = 130\ MPa\sqrt{m}$, $\alpha = 1/3$, $n = 3$, $R = 0.7$ and $A = 5.11x10^{-11}\ \dfrac{MN^n \cdot mm^{\frac{2n+2}{2}}}{cycles}$, then $n(1-\alpha) = (3)(1-1/3) = 2$

Thus,

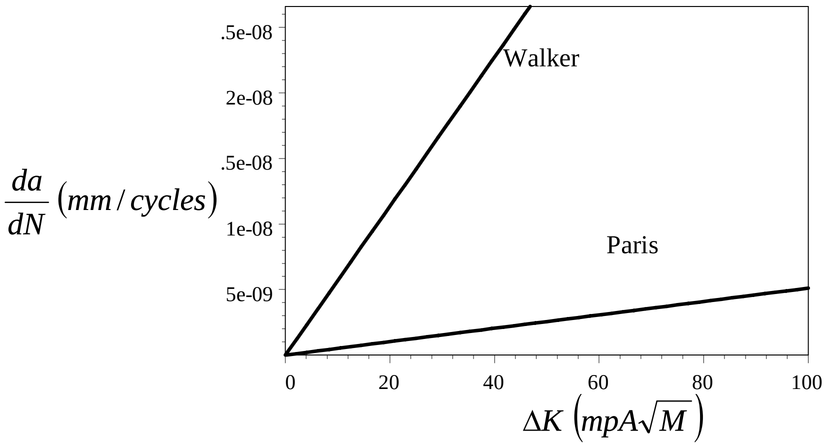
$$C = \frac{A}{(1-R)^{n(1-\alpha)}} = \frac{5.11x10^{-11}\ MN^3 \cdot mm^4 / cycles}{(1-0.7)^2} = 5.68x10^{-10}\ MN^3 \cdot mm^4 / cycles$$

Comparison:

Walker: $\dfrac{da}{dN} = \left(5.68 \times 10^{-10}\ \dfrac{MN^3 \cdot mm^4}{cycles}\right)(\Delta K)^3$

Paris: $\dfrac{da}{dN} = \left(5.11 \times 10^{-11}\ \dfrac{MN^3 \cdot mm^4}{cycles}\right)(\Delta K)^3$

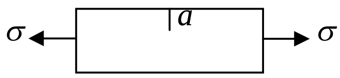
Plot $\dfrac{da}{dN}$ $vs.\ \Delta K$ for $20\ MPa\sqrt{m} \le \Delta K \le 100\ MPa\sqrt{m}$



9.3 Suppose that a single-edge crack in a plate grows from 2 mm to 10 mm at a constant loading frequency of 0.02 Hz. The applied stress ratio and the maximum stress are zero and 403 MPa, respectively. The material has a plane strain of 80 MPa.m$^{1/2}$ and a crack growth behavior described by

$$\frac{da}{dN} = 3.68x10^{-12}\left(\Delta K_I\right)^4$$

Here, da/dN is in m/cycle and ΔK_I is in $MPa\sqrt{m}$. Determine the time it takes for rupture to occur.



Solution:

$K_{IC} = 80\ MPa\sqrt{m}$ $a_o = 2\ mm$ $a_f = 10\ mm$ $R = 0$ $\alpha = 1.12$ $f = 20\ Hz$

$A = 3.68x10^{-12}$ $n = 4$ $\sigma_{min} = 0$ $\sigma_{\max} = 403\ MPa$ $\Delta\sigma = 403\ MPa$

Using $K_{IC} = \alpha\sigma_{max}\sqrt{\pi a_c}$ yields

$$a_c = a_f = 10\ mm$$

$$\frac{da}{dN} = A(\Delta K_I)^n = A\left[\alpha\Delta\sigma\sqrt{\pi a}\right]^4$$

$$\int_{a_o}^{a_f} a^{-2} da = A\left[\alpha\Delta\sigma\sqrt{\pi}\right]^4 \int_0^{N_f} dN$$

$$a_o^{-1} - a_f^{-1} = 1.51 N_f$$

$$N_f = \frac{\left(2x10^{-3}\ m\right)^{-1} - \left(10x10^{-3}\ m\right)^{-1}}{1.51\ m^{-1}.cycles^{-1}}$$

$$N_f = 278.31\ cycles$$

$$t = \frac{N_f}{f} = \frac{278.31\ cycles}{0.020\ cycles/\sec} = 13{,}916 \sec$$

$$t = 3.87\ hours$$

9.4 If a large component is subjected to a cyclic loading under $\Delta\sigma = 300\ MPa$ and $R = 0$. The material behaves according to Paris Law

$$\frac{da}{dN} = 2x10^{-8}(\Delta K_I)^{2.45}$$

where da/dN is in m/cycles and ΔK_I is in MPa $MPa\sqrt{m}$. Determine the plane strain fracture toughness for the component to endure 37,627 cycles so that a single-edge crack grows from 2 mm to a_c.

Solution:

$$\frac{da}{dN} = 2x10^{-8}(\Delta K_I)^{2.45} = A\left(1.12\Delta\sigma\sqrt{\pi}\right)^{2.45} a^{2.45}$$

$$\int_{a_o}^{a_c} a^{-2.45} da = 0.12577 \int_0^{N_f} dN = 4{,}732.30$$

$$\frac{a_o^{-1.45} - a_c^{-1.45}}{1.45} = 4{,}732.30$$

$$a_c = a_f \approx 7\ mm$$

$$K_{IC} = 1.12\Delta\sigma\sqrt{\pi a_c}$$

$$K_{IC} = (1.12)(300\ MPa)\sqrt{\pi\left(7x10^{-3}\ m\right)}$$

$$K_{IC} \approx 50\ MPa\sqrt{m}$$

9.5 Consider a part made of a polycrystalline metal that is stresses in the elastic stress range. If the metal contains inclusions, has an imperfectly smooth exterior surface, and natural dislocation, would the metal experience irreversible changes in a micro-scale? Explain.

Answer: Many dislocations would move and contribute to a small irreversible process since the dislocation configurations would eventually change. This is one mechanism would lead to fatigue cracking due to dislocations.

9.6 Why most service fatigue fractures are normally not clear?

Answer: Most service fatigue fractures experience varying stress amplitude, leading to irregularly spaced striations, if any.

9.7 What is the physical meaning of the slope n of the Stage II line according to the Paris model?

Answer: According to Paris law, $\dfrac{da}{dN} = A(\Delta K_I)^n$, the higher the value of the slope n, the higher the crack growth rate da/dN. For instance, if $\Delta K > 0$ and $n \to \infty$, then $(\Delta K_I)^n \to \infty$ and $da/dN \to \infty$ since A is a constant.

9.8 Suppose that $d(2a)/dN = 0.001\, mm/cycle$ and $n = 4$ in the Paris equation for 7075-T6 (FCC), 2024-T3 (FCC), Mo (BCC), and steel (BCC). Determine a) the constant A and its units, and b) which of these materials will have the higher crack growth, rate?

Solution:

From Figure 9.6,

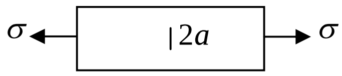
$$\frac{d(2a)}{dN} = A(\Delta K_I)^4 \quad \text{and} \quad A = \frac{10^{-3}\ mm/cycle}{(\Delta K_I)^4}$$

Material	$\Delta K_I \ \left(MPa\sqrt{m}\right)$	$A \ \left(MN^{-4}\cdot m^5/cycle\right)$
7075-T6 (FCC)	10	$1.00x10^{-10}$
2024-T3 (FCC)	15	$1.98x10^{-11}$
Mo (BCC)	20	$6.25x10^{-12}$
Steel (BCC)	30	$1.23x10^{-12}$

Therefore, 7075-T6 (FCC) alloy has the highest fatigue crack growth rate because it has the highest constant A.

9.9 A Ti-6Al-4V large plate containing a 4-mm long central crack is subjected to a steady cyclic loading R = 0.10. The plate plane strain and the threshold fracture toughness are 70 $MPa\sqrt{m}$ and 14.7 $MPa\sqrt{m}$. Determine **(a)** the minimum stress range, **(b)** the maximum applied stress range for a fatigue life of 3,000 cycles, and **(c)** the critical crack size for 3,000 cycles. Let the Paris equation be applicable so that $n = 4$ and $A = 10^{-12}\ MN^{-4} \cdot m^{12} / cycles$.

Solution:



$2a_o = 4\ mm$ $\quad N = 10^3\ cycles$ $\quad K_{IC} = 70\ MPa\sqrt{m}$ $\quad \alpha = 1$

$a_o = 2\ mm$ $\quad R = 0.40$ $\quad K_{th} = 14.7\ MPa\sqrt{m}$

(a) Using eq. (9.2b) yields

$$\Delta\sigma_{th} = \frac{\Delta K_{th}}{\alpha\sqrt{\pi a_o}} = \frac{14.7\ MPa\sqrt{m}}{\sqrt{\pi\left(2x10^{-3}\ m\right)}} = 185.45\ MPa$$

$$\Delta\sigma_{min} < \Delta\sigma_{th} = 185.45\ MPa$$

(b) From eq. (9.9) with $N_o = 0$ since a_o exists,

$$C_1\left(\Delta\sigma\right)^n + C_2\left(\Delta\sigma\right)^{n-2} - C_3 = 0 \qquad (n \neq 2)$$

$$C_1 = A\left(N - N_o\right)\left(n-2\right)\left(K_{IC}\right)^n = 0.50\ m$$

$$C_2 = (2/\pi)(K_{IC}/\alpha)^2 = 3.12x10^3\ MPa^2 \cdot m$$

$$C_3 = (2a_o^{1-n/2})(1/\pi)^{n/2}(K_{IC}/\alpha)^n = 2.43x10^9\ MPa^2 \cdot m$$

$$0.50\Delta\sigma^4 + 3.12x10^3\Delta\sigma^2 - 2.43x10^9 = 0$$

Thus,

$$\Delta\sigma^2 = -\frac{C_2}{2C_1} \pm \frac{1}{2C_1}\sqrt{C_2^2 - 4C_1C_3} = -3.12x10^4\ MPa^2 \pm 3.81x10^5\ MPa^2$$

$\Delta\sigma \approx 642\ MPa = \Delta\sigma_{max}$ (Positive real root)

(c) $R = \dfrac{\sigma_{min}}{\sigma_{max}}$ and $\Delta\sigma = \sigma_{max} - \sigma_{min}$

$R\sigma_{max} = \sigma_{min}$ and $\sigma_{min} = \sigma_{max} - \Delta\sigma$

$$R\sigma_{max} = \sigma_{max} - \Delta\sigma$$

$$\sigma_{max} = \frac{\Delta\sigma}{1-R} = \frac{642.31\ MPa}{1-0.1} \approx 713\ MPa$$

Then,

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{\alpha\sigma_{max}}\right)^2 = 3.1\ mm \text{ and } 2a_c = 6.2\ mm$$

9.10 Plot $da/dN \ vs. \ \Delta K$ for 403 S.S. using the Paris, Forman and Broek/Schijve equations. Use the data given in Table 9.2 and a (20 mm)x(300 mm)x(900 mm) plate containing a single-edge crack of 2-mm long. Let $20 \ MPa\sqrt{m} \le \Delta K \le 80 \ MPa\sqrt{m}$.

Solution:

$a = 2x10^{-3} \ m$ $\quad B = 20 \ mm$ $\quad w = 300 \ mm$ $\quad L = 900 \ mm$ and

$a/w = 0.007$ and $\quad \alpha = f(a/w) \approx 1.12$

Using data for AISI 403 stainless steel yields

$K_{IC} = 36 \ MPa\sqrt{m}$ $\quad R = \sigma_{min}/\sigma_{max}$ and $\quad \Delta\sigma = \sigma_{max} - \sigma_{min}$

$\Delta K_{th} = 1.37 \ MPa\sqrt{m}$ $\quad R\sigma_{max} = \sigma_{min}$ and $\quad \sigma_{min} = \sigma_{max} - \Delta\sigma$

$\sigma_{ys} = 1172 \ MPa$ $\quad R\sigma_{max} = \sigma_{max} - \Delta\sigma$

$R = 0.1$ $\quad n = 3.5$ $\quad \sigma_{max} = \Delta\sigma/(1-R)$ for $R < 1$

$A = 5.98x10^{-13}$

Need to calculate K_{max} for each given ΔK. For instance, using the upper limit in the range $20 \ MPa\sqrt{m} \le \Delta K \le 80 \ MPa\sqrt{m}$, the procedure is as follows:

$$K_{max} = \alpha\sigma_{max}\sqrt{\pi a} = \frac{\alpha\Delta\sigma}{1-R}\sqrt{\pi a} \text{ and } \Delta K = \alpha\Delta\sigma\sqrt{\pi a} = 80 \ MPa\sqrt{m}$$

Solve for the stress range,

$\Delta\sigma = 901.12 \ MPa$ (Maximum)

$\longrightarrow \sigma_{max} = \Delta\sigma/(1-R) \approx 1001 \ MPa$ (upper limit)

$\longrightarrow K_{max} = \alpha\sigma_{max}\sqrt{\pi a} \approx 89 \ MPa\sqrt{m}$ (upper limit)

$$\longrightarrow a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{\alpha\sigma_{max}}^2\right) = 48.30 \ mm$$

It is suggested that you use a computer program to handle all the calculations needed in order to plot smooth curves.

Paris : $$\frac{da}{dN} = A(\Delta K)^n \qquad (9.6)$$

Forman : $$\frac{da}{dN} = \frac{A(\Delta K)^n}{(1-R)K_{IC} - \Delta K} \quad (Plane\ strain) \qquad (9.10)$$

Broek/Schijve : $$\frac{da}{dN} = AK_{\max}^2 \cdot \Delta K^n \qquad (9.11)$$

9.11 Plot the data given below and use eq. (9.6) as the model to draw a curve fitting line on log-log scales. Determine the constants in such an equation.

No.	ΔK $\left(MPa\sqrt{m}\right)$	da/dN $\left(m/cycle\right)$
1	20	2.14x10^{-8}
2	30	8.84x10^{-8}
3	40	2.42x10^{-7}
4	50	5.29x10^{-7}
5	60	1.00x10^{-6}
6	70	1.72x10^{-6}
7	80	2.74x10^{-6}

Solution:
Since the Stage II data is for a linear behavior in a log-log scale, the constants in eq. (9.6) can be estimated as follows:

$$\left(\frac{da}{dN}\right)_1 = A(\Delta K_1)^n \quad \text{and} \quad \left(\frac{da}{dN}\right)_2 = A(\Delta K_2)^n$$

$$\frac{\left(\frac{da}{dN}\right)_1}{\left(\frac{da}{dN}\right)_2} = \left(\frac{\Delta K_1}{\Delta K_2}\right)^n$$

$$n = \frac{ln\left[(da/dN)_1/(da/dN)_2\right]}{ln\left[\Delta K_1/\Delta K_2\right]} = \frac{ln\left[2.14x10^{-8}/8.84x10^{-8}\right]}{ln\left[20/30\right]}$$

$$n = 3.50$$

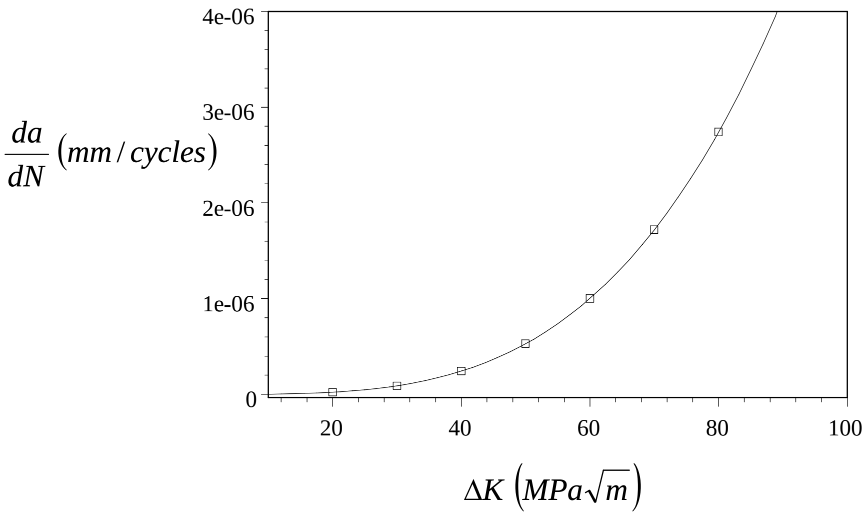
$$A = \frac{(da/dN)_1}{(\Delta K_1)^n} = \frac{2.14x10^{-8}\ m/cycles}{\left(20\ MPa\sqrt{m}\right)^{3.50}}$$

$$A = 5.98x10^{-13}\ \ MPa^{-3.50} \cdot m^8/cycles$$

Then, the equation for curve fitting is of the following nature

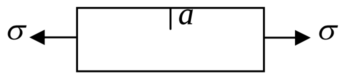
$$\frac{da}{dN} = 5.98x10^{-13}\ (\Delta K)^{3.50}$$

Perhaps, the curve fitting procedure gives slightly different results.



9.12 A steel plate containing a single edge crack was subjected to a uniform stress range $\Delta\sigma$ at a stress ratio of zero. Fatigue fracture occurred when the total crack length was 0.03 m. Subsequent fatigue failure analysis revealed a striation spacing per unit cycle of 7.86x10^{-8} m. The hypothetical steel has a modulus of elasticity of 207 GPa. Predict **(a)** the maximum cyclic stress for a crack length of 0.01 m**, (b)** the striation spacing per unit cycle when the crack length is 0.02 m, **(c)** the Paris equation constants and (d) the plane strain fracture toughness and e) the fatigue crack growth rate assuming that the Paris equation is applicable nearly up to fracture.

Solution:



(a) $x = 7.86x10^{-8}\ m$

$$\left(\frac{da}{dN}\right)_a \approx \frac{\Delta a}{\Delta N} = 7.86x10^{-8}\ m/cycles$$

$$\Delta K_a = E\sqrt{\frac{x}{6}} = (207{,}000\ MPa)\sqrt{\frac{7.86x10^{-8}\ m}{6}} = 23.69\ MPa\sqrt{m} = K_{\max}$$

Thus,

$$\Delta\sigma = \frac{\Delta K}{\sqrt{\pi a}} = \frac{23.69\ MPa\sqrt{m}}{\sqrt{\pi(0.01\,m)}} = 133.66\ MPa = \sigma_{\max}$$

(b) For $a = 0.02\,m$,

$$\Delta K_b = K_{\max} = \Delta\sigma\sqrt{\pi a} = (133.69\ MPa)\sqrt{\pi(0.02\,m)} = 33.50\ MPa\sqrt{m}$$

Then,

$$x = 6\left(\frac{\Delta K}{E}\right)^2 = 6\left(\frac{33.50\ MPa\sqrt{m}}{207{,}000\ MPa}\right)^2 = 1.57x10^{-7}\ m$$

$$\left(\frac{da}{dN}\right)_b \approx \frac{\Delta a_b}{\Delta N} = \frac{x}{\Delta N} = \frac{1.57x10^{-7}\ m}{1\,cycle} = 1.57x10^{-7}\ m/cycle$$

(c) The exponent in the Paris equation is

$$\left.\begin{aligned}\left(\frac{da}{dN}\right)_a &= A(\Delta K_a)^n\\ \left(\frac{da}{dN}\right)_b &= A(\Delta K_b)^n\end{aligned}\right. \longrightarrow \quad n = \frac{\ln\left[\frac{(da/dN)_a}{(da/dN)_b}\right]}{\ln\left[\frac{\Delta K_a}{\Delta K_b}\right]} = \frac{\ln\left[\frac{7.86x10^{-8}}{1.57x10^{-7}}\right]}{\ln\left[\frac{23.69}{33.50}\right]} = 2$$

and the constant is

$$A = \frac{(da/dN)_a}{(\Delta K_a)^n} = \frac{7.86x10^{-8}\ m/cycle}{\left(23.69\ MPa\sqrt{m}\right)^2} = 1.40x10^{-10}\ MPa^{-2}/cycle$$

$$\frac{da}{dN} = \left(1.40x10^{-10}\right)(\Delta K)^2$$

(d) For a_c = 0.03 m,

$$K_{IC} = \sigma_{\max}\sqrt{\pi a_c} = (133.66\ MPa)\sqrt{\pi(0.03\ m)}$$

$$K_{IC} \approx 41\ MPa\sqrt{m}$$

(e) Finally,

$$A = \frac{(da/dN)_a}{(\Delta K_a)^n} = \frac{7.86x10^{-8}\ m/cycle}{\left(23.69\ MPa\sqrt{m}\right)^2} = 1.40x10^{-10}\ MPa^{-2}/cycle$$

$$\frac{da}{dN} = \left(1.40x10^{-10}\right)(K_{IC})^2 = \left(1.40x10^{-10}\right)(41)^2 = 2.35x10^{-7}\ m/cycle$$

9.13 A 2-cm thick pressure vessel made of a high strength steel welded plates burst at an unknown pressure. Fractographic work using a scanning electron microscope (SEM) revealed a semielliptical fatigue surface crack (a = 0.1 cm and 2c = 0.2 cm) located perpendicular to the hoop stress and nearly in the center of one of the welded plates. The last fatigue band exhibited three striations having an average length of 0.34 mm at 10,000 magnification. The vessel internal diameter was 10 cm. Calculate a) the pressure that caused fracture and b) the time it took for fracture to occur due to pressure fluctuations. Assume a pressure frequency of 0.1 cycles per minute (cpm). Given data: σ_{ys} = 600 MPa, E = 207 GPa, $K_{IC} = 75\ MPa\sqrt{m}$ and

$$da/dN = 4.50x10^{-7}(\Delta K)^2.$$

Solution:

$$\frac{a}{2c} = \frac{0.1}{0.2} = 0.5$$

$$\frac{a}{B} = \frac{0.1}{2} = 0.05$$

$$\frac{da}{dN} \approx \frac{\Delta a}{\Delta N} = \frac{0.00034\ m}{3\ cycle} = 1.13x10^{-4}\ m/cycle$$

Using the given crack growth rate equation, ΔK becomes

$$\Delta K = \left(\frac{da/dN}{A} \right)^{1/2} = \left(\frac{1.13x10^{-4}\ m/cycle}{4.50x10\ MPa^{-2}/cycle} \right)^{1/2}$$
$$\Delta K = 15.85\ MPa\sqrt{m}$$

From eq. (9.23),
$$\Delta K = E\sqrt{\frac{x}{6}} = (207{,}000\ MPa)\sqrt{\frac{0.00034\,m}{(6)(10{,}000)}}$$
$$\Delta K = 15.58\ MPa\sqrt{m}$$

which is similar to the previous ΔK value. The average becomes $\Delta K = 15.72\ MPa\sqrt{m}$. Assume the following stress ratio $\sigma/\sigma_{ys} = 0.64$ so that $\sigma = 384\ MPa$, which is verified below. Thus, the shape factor becomes

$$Q = \left(\frac{\pi}{2}\right)^2 \left[\frac{3}{4} + \left(\frac{a}{2c}\right)^2\right]^2 - \frac{7}{33}\left(\frac{\sigma}{\sigma_{ys}}\right)^2 = \left(\frac{\pi}{2}\right)^2 \left[\frac{3}{4} + \left(\frac{0.1}{0.2}\right)^2\right]^2 - \frac{7}{33}(0.64)^2 = 2.38$$

The hoop stress range along with the average value of ΔK is

$$\Delta\sigma = \frac{\Delta K}{\alpha\sqrt{\pi a/Q}} = \frac{15.72\ MPa\sqrt{m}}{1.12\sqrt{\pi(0.001\,m)/2.38}} = 386.32\ MPa$$

Thus,

$$\frac{\Delta\sigma}{\sigma_{ys}} = \frac{386.32}{600} \approx 0.64$$

Therefore, the assumed stress ratio is reasonably correct and fortunately, a single iteration is sufficient. The pressure is estimated as

$$P = \frac{2B\sigma}{d} = \frac{(2)(2\ cm)(384\ MPa)}{10\ cm} = 154.40\ MPa$$

(b) The critical crack length is
$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{\alpha\Delta\sigma}\right)^2 = \frac{1}{\pi}\left(\frac{75\ MPa\sqrt{m}}{1.12x386.32\ MPa}\right)^2 = 9.56x10^{-3}\ m = 0.956\ cm$$

Integrating eq. (9.15) along with $N_o = 0$ yields

$$\int_{N_o}^{N} dN = \int_{a_o}^{a_c} \frac{da}{A(\Delta K)^2} = \int_{a_o}^{a_c} \frac{da}{A\left(\alpha\Delta\sigma\sqrt{\pi a}\right)^2} = \frac{1}{\pi A(\alpha\Delta\sigma)^2}\int_{a_o}^{a_c} \frac{da}{a} = \frac{\ln(a_c / a_o)}{\pi A(\alpha\Delta\sigma)^2}$$

$$N = \frac{\ln(a_c / a_o)}{\pi A(\alpha\Delta\sigma)^2} = \frac{\ln(0.956/0.1)}{\pi\left(4.50x10^{-7}\right)(1.12x386.32)^2} = 8.53\,cycles$$

Then,

$$t = \frac{N}{cpm} = \frac{8.53\,cycles}{0.1\,cycle/\min} = 85.30\min$$

9.14 Show that $a_c = a(K_{IC} / K_{\max})^2$

Solution:

From,

$K_{IC} = \alpha\sigma_{\max}\sqrt{\pi a_c}$ and $K_{\max} = \alpha\sigma_{\max}\sqrt{\pi a}$

$$\frac{K_{IC}}{K_{\max}} = \frac{\alpha\sigma_{\max}\sqrt{\pi a_c}}{\alpha\sigma_{\max}\sqrt{\pi a}} = \frac{\sqrt{a_c}}{\sqrt{a}}$$

$$\left(\frac{K_{IC}}{K_{\max}}\right)^2 = \frac{a_c}{a}$$

Then,

$$a_c = a\left(\frac{K_{IC}}{K_{\max}}\right)^2$$

CHAPTER 10

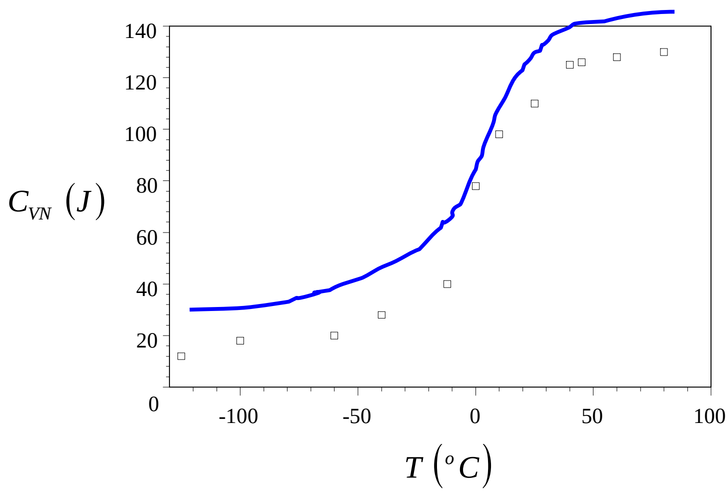
FRACTURE TOUGHNESS CORRELATIONS

10.1 **(**a) Plot the given C_{VN}–T data for a carbon steel. **(b)** Calculate K_{IC} using the C_{VN} values up to zero ^oC and Plot K_{IC}-cal and K_{IC}-exp. vs. Temperature. Is there a significant difference? If so, explain.

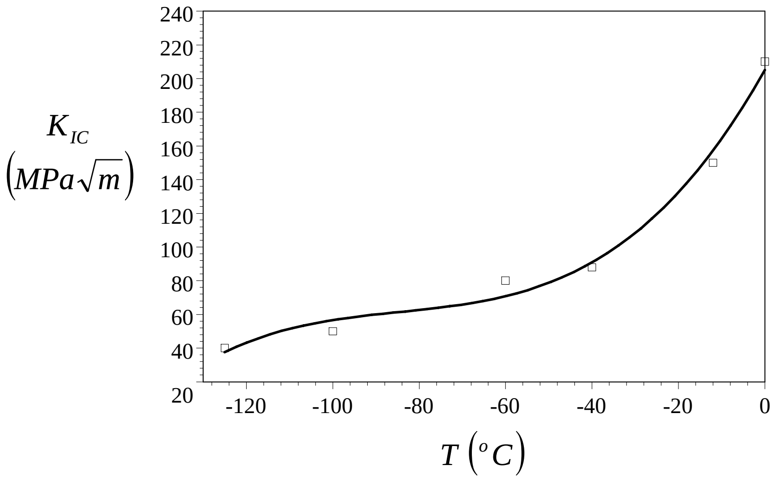
T (^oC)	-125	-100	-60	-40	-12	0	10	25	40	45	60	80
C_{VN} (J)	12	18	20	28	40	78	98	110	125	126	28	30
K_{IC}-exp. (MPa.m$^{1/2}$)	40	50	80	88	50	210						

Solution:

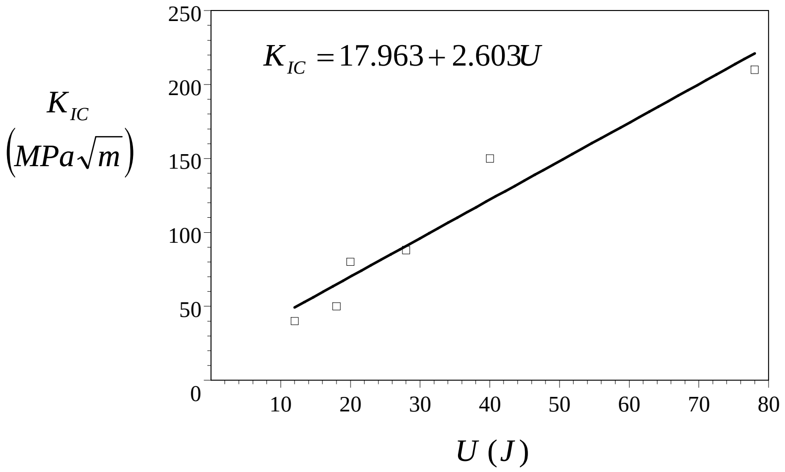
(a) Plots

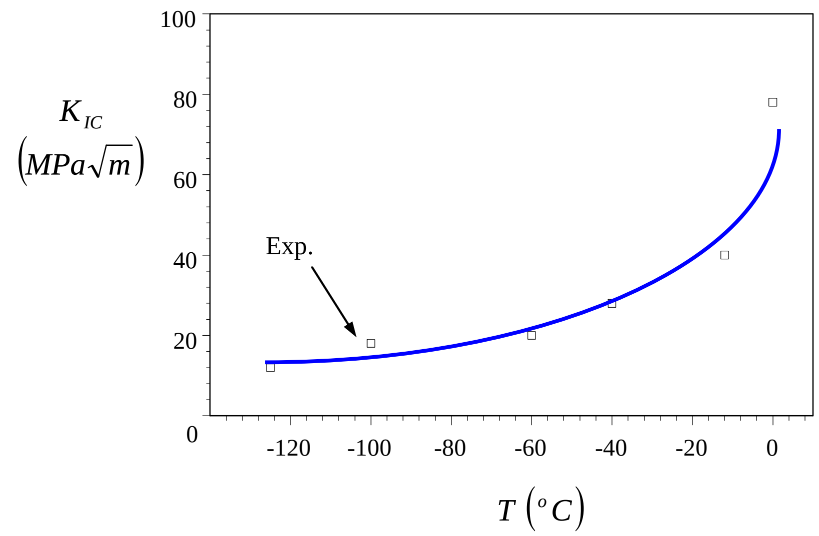


$$K_{IC} = 205.05 + 4.5721T + 5.0958x10^{-2}T^2 + 2.0082x10^{-4}T^3$$



(b) Do nonlinear curve fitting on K_{IC} and C_{VN} data set as per eqs. (10.66) and (10.67). Then, use eq. (10.68) for calculating K_{IC} values. Thus, plot K_{IC}-cal and K_{IC}-exp. vs. Temperature. Below is a linear curve fitting result for plane strain fracture toughness and Charpy impact energy.





10.2 A mild steel plate has a through the thickness single-edge crack, a yield strength of 800 MPa and a static fracture strength is $\sigma_f = 3.2\sigma_{ys}$. If the plate is loaded in tension and fractures at 600 MPa, calculate the plane strain fracture toughness of the steel plate and the critical crack length.

Solution:

$\sigma_f = 3.2\sigma_{ys}$ and $\sigma_{ys} = 800\ MPa$

$\lambda = \dfrac{\sigma_f}{\sigma_{ys}} = 3.2$ and $\beta = 20\ m^{-1/2}$

From eq. (10.12),

$$K_{IC} = \frac{(\lambda - 1)\sigma_{ys}}{\beta} = \frac{(3.2 - 1)(800\ MPa)}{20\ m^{-1/2}} = 88\ MPa\sqrt{m}$$

If $\sigma = 600\ MPa,$ then

$$K_{IC} = 1.12\sigma\sqrt{\pi a_c}$$

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{1.12\sigma}\right)^2 = \frac{1}{\pi}\left[\frac{88\ MPa\sqrt{m}}{(1.12)(600\ MPa)}\right]^2 = 5.46\ mm$$

10.3 A standard Charpy specimen with $B = 5\ cm$, $w = 5\ cm$ and $S/w = 4$ was tested at a room temperature. The measured impact energy was 30 joules. The tested material has a modulus of elasticity of 70,000 MPa. Calculate the K_{IC} for this hypothetical specimen having $x = a/w = 0.2$.

Solution:

Data: $x = a/w = 0.2$ $w = 5\times10^{-2}\ m$ $B = 5\times10^{-2}\ m$

$E = 70{,}000\ MPa$ $U = 30\times10^{-6}\ MJ = 30\times10^{-6}\ MN.m$

From eqs. (10.57) and (10.43), respectively,

$$\phi = \frac{2}{7\pi x} = \frac{2}{7\pi(0.2)} = 0.45473$$

$$K_{IC} = K_I = \sqrt{\frac{UE}{\phi Bw}} = \sqrt{\frac{(30x10^{-6}\ MN.m)(70{,}000\ MPa)}{(0.45473)(5x10^{-2}\ m)(5x10^{-2}\ m)}}$$

$$K_{IC} == 42.98\ MPa\sqrt{m}$$

10.4 Suppose that a design code calls for a Charpy impact energy of 22 Joules for building a large pressure vessel containing an inert gas. If A533B and A723 ($\sigma_{ys} = 1{,}100\ MPa$) steel plates are available for such a purpose, then **(a)** select the steel that will tolerate the largest critical crack length (depth of a surface semi-elliptical crack) when the hoop stress is 500 MPa and **(b)** determine the minimum plate thickness as per ASTM E399 standard for the selected steel.

Solution:

For $\sigma = 500\ MPa,\ U = 22\ J$ and $\sigma_{ys} = 1{,}100\ MPa$ (A723),

$$K_{IC}(\text{A533B}) = 20+139\left(\frac{196)}{U}\text{-}1\right)^{-0.54} = 65.50\ MPa\sqrt{m} \qquad \text{eq. (10.71)}$$

$$K_{IC}(\text{A723}) = \sqrt{0.644U\sigma_{ys} - 0.006\sigma_{ys}^2} = 91.24\ MPa\sqrt{m} \qquad \text{eq. (10.74)}$$

(a) The critical length

$$a_c(\text{A533B}) = \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma}\right)^2 = \frac{1}{\pi}\left(\frac{65.50\ MPa\sqrt{m}}{500\ MPa}\right)^2 = 5.46\ mm$$

$$a_c(\text{A723}) = \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma}\right)^2 = \frac{1}{\pi}\left(\frac{91.24\ MPa\sqrt{m}}{500\ MPa}\right)^2 = 10.60\ mm$$

The ASTM A723 steel allows a larger critical crack length; therefore, this material is selected for such an application.

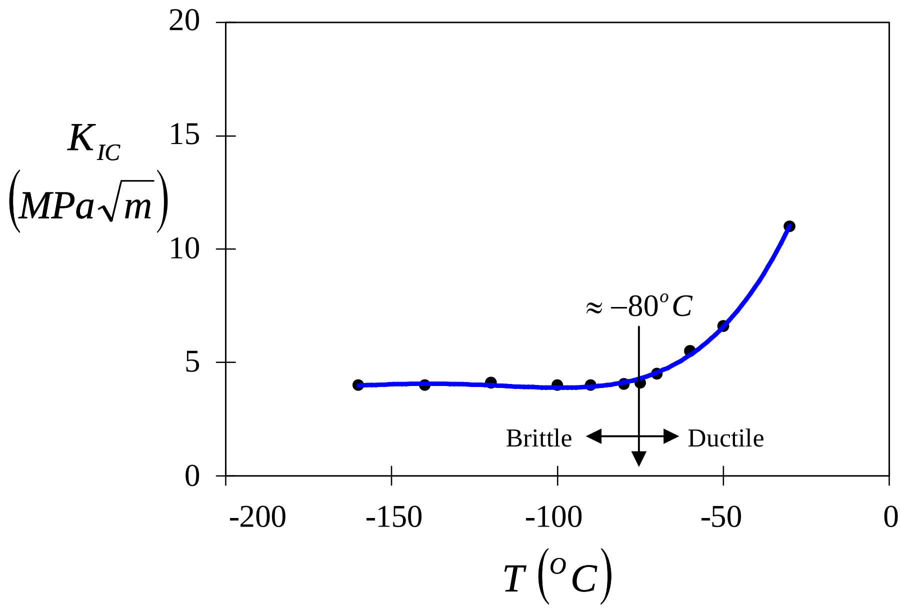
(b) The plate thickness for the selected A723 steel is

$$B \geq 2.5\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = 2.5\left(\frac{91.24\ MPa\sqrt{m}}{1{,}100\ MPa}\right)^2 = 17.20\ mm \approx 0.6772\ inches$$

10.5 Plot the fracture toughness data for a hypothetical polymer and determine the brittle-ductile transition temperature.

T (°C)	-160	-120	-90	-80	-75	-70	-60	-50	-30
K_{IC} $(MPa\sqrt{m})$	4.10	4.11	4.00	4.05	4.10	4.50	5.50	6.60	11.00

Solution:



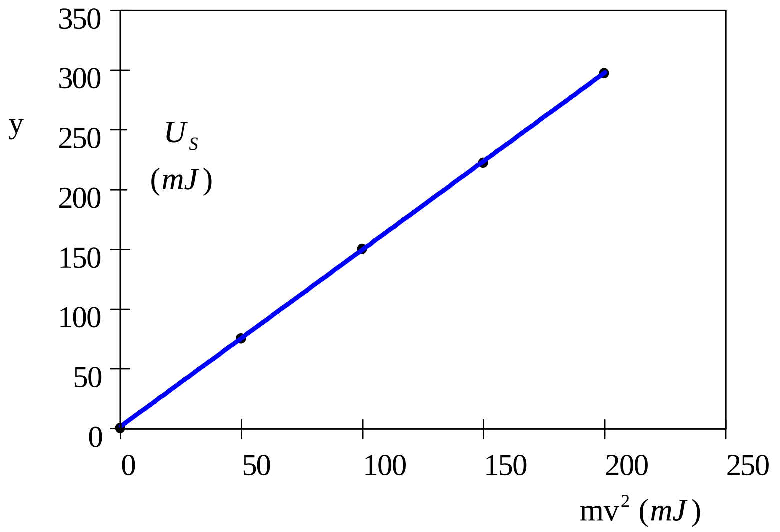
10.6 For linear motion during Charpy impact tests, the energy lost by the striker (U_S) and the kinetic energy (mv^2) data for some polymers with different masses and spans to depth (S/w) ratios [47] are given below. Let $m/M << 1$ and a) plot $U_S = f(mv^2)$ and estimate the coefficient of restitution (e) for these data. What does $0 < e < 1$ mean? Theoretically, determine b) the $U_S = f(mv^2)$ ratio for the first bound which is transformed into specimen strain energy and c) the $U_S = f(mv^2)$ ratio when there is no bouncing.

mv^2 (mJ)	0	50	100	150	200
U_S (mJ)	0	75	150	222	297

Solution:

(a) The plot and the coefficient of restitution are

Polynomial fit: $\boxed{y = 0.6 + 1.482x \text{ or } U_S = 0.6 + 1.482(mv^2)}$



From eq. (10.39) along with $m/M << 1$,

$$U_S = (1+e)mv^2$$
$$Slope = 1+e = 1.5$$
$$e = 0.5$$

This result, $e = 0.5$, means that a repeated impact of converging energy into strain energy occurs until the last contact is achieved.

(b) The $U_S = f(mv^2)$ ratio for the first bound which is transformed into specimen strain energy requires that the coefficient of restitution be $e = 1$. Thus,

$$U_S/(mv^2) = 1+e = 2$$

c) The $U_S = f(mv^2)$ ratio when there is no bouncing means that $e = 0$. Thus,

$$U_S/(mv^2) = 1+e = 1$$

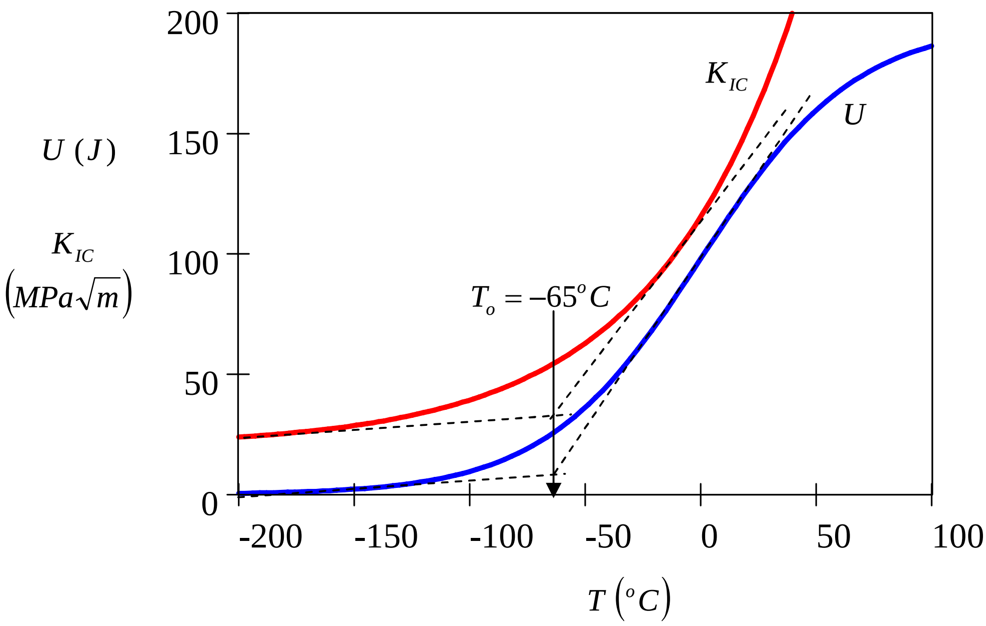
10.7 A large and thick ASTM A533B-1 steel plate containing a 4-mm long through-the-thickness central crack fractures when it is subjected to a tensile stress of 7 MPa. Plot U, K_{IC}, and G_{IC} at $-200^oC \le T \le 100^oC$. Which of the plots is more suitable for determining the transition temperature? Explain. Data: E = 207,000 MPa and ν = 1/3.

Solution:

From eq. (10.72) and (10.73),

$$U = \frac{196}{1+\exp(-0.0297T)} \quad \text{in Joules}$$

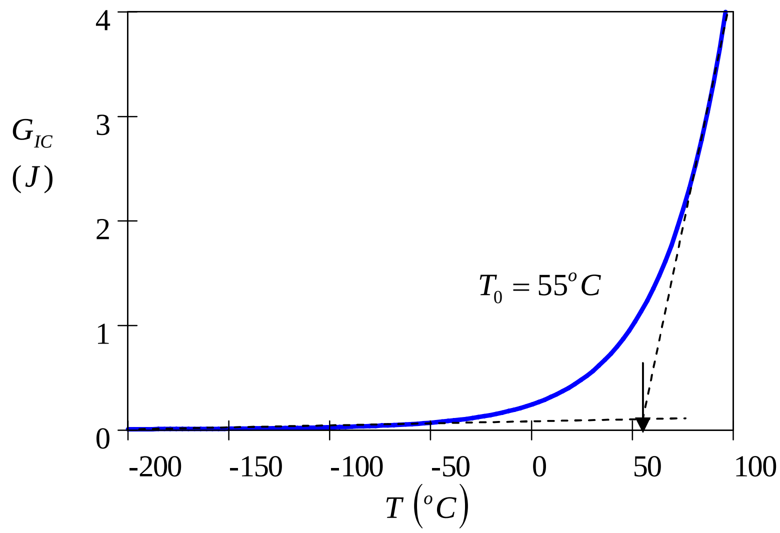
$$K_{IC} = 20 + 139\left(\frac{196}{U} - 1\right)^{-0.54} \quad \text{in } MPa\sqrt{m}$$



Denote that the transition temperature (T_o) is easily determined using the $U = f(T)$ plot. This is shown with dash lines.

$$G_{IC} = \frac{(1-\nu^2)}{E}K_{IC}^2 = \frac{(10^6)\left[1-(1/3)^2\right]}{207{,}000}\left[20+95.50\exp(0.016T)\right] \quad \text{in Joules}$$

$$G_{IC} = (4.2941)\left[20+95.50\exp(0.016T)\right]$$



It is clearly shown that the $U_{IC} = f(T)$ plot is more suitable for determining the transition temperature $T_0 = -65^oC$.

10.8 A large plate made of 18Ni-8Co-3Mo Grade 200 alloy is part of structure exposed to relatively high temperature. Charpy impact tests were carried out and the average impact energy is 60 J. Use this information to calculate (a) the plane strain fracture toughness, (b) the minimum thickness ASTM requirement. The plate width is at least twice the thickness. Is this thickness practical? (c) Assume that a single-edge through the thickness crack develops. What will the critical crack length be? Will its value be reasonable?

Solution:

(a) If $= 60\ J$ and $\sigma_{ys} = 1{,}310\ MPa$ (From Table 10.1), then eq. (10.76) yields

$$\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = 4.69\left(\frac{U}{\sigma_{ys}}\right) - 0.20$$

$$K_{IC} = \sigma_{ys}\sqrt{4.69\left(\frac{U}{\sigma_{ys}}\right) - 0.20}$$

$$K_{IC} = \left(1{,}310\ MPa\right)\sqrt{4.69\left(\frac{60}{1{,}310}\right) - 0.20}$$

$$K_{IC} = 159.42\ MPa\sqrt{m}$$

(b) From eq. (3.30), the minimum plate thickness is

$$B \geq 2.5\left(\frac{K_{IC}}{\sigma_{ys}}\right)^2 = (2.5)\left(\frac{159.42}{1310}\right)$$
$$B \geq 0.30424\ m = 304.24\ mm \simeq 12\ inches$$

This is a very impractical thick for plate, but it complies with the ASTM plane strain conditions.

(c) Now if a single-edge crack develops during service, the critical crack length under a stress half the yield strength will be

$$K_{IC} = \frac{\sigma_{ys}}{2}\sqrt{\pi a_c}$$
$$a_c = \frac{1}{\pi}\left(\frac{2K_{IC}}{\sigma_{ys}}\right)^2 = \frac{10^3}{\pi}\left(\frac{2 \times 159.42}{1310}\right)^2$$
$$a_c \simeq 19\ mm$$

This is a reasonable result for a plate so thick.